European Journal of Applied Sciences – Vol. 13, No. 05
Publication Date: September 29, 2025
DOI:10.14738/aivp.1305.19428.
Tong, M. & Ton, D. (2025). The Intrinsic Topologies of Partially Ordered Sets. *European Journal of Applied Sciences, Vol - 13(05).* 210-219.

The Intrinsic Topologies of Partially Ordered Sets

Maoting Tong
Public Basic Department, Nanjing
Vocational University of Industry Technology,
Nanjing, 210023, P.R. China

Daorong Ton
Department of Mathematics,
Hohai University, Nanjing, 210098, P.R. China

ABSTRACT

In this article, we discuss the relationship and equivalence between the intrinsic topologies of a partially ordered set. The main results are Theorem 1-Theorem 9. The interval topology is coarser than the open interval topology in a partially ordered set. The order topology is coarser than the open interval topology in a lattice. In a partially ordered set P with finite divergence, $i = L$ if and only if the limit of each L-topological convergemt net in P is midpoint. Mathematics Subject Classification (2010). Primary 06F15, Secondary 20F60

Keywords: Partially ordered set, Intrinsic topology, Sequential convergence, Finite divergence.

THE INTRINSIC TOPOLOGIES OF A PARTIALLY ORDERED SET

We discussed the intrinsic topology of lattice groups in [1], and various intrinsic topologies can also be defined in an ordered relationship within a partially ordered set P.

In this section, we present the relationships between these intrinsic topologies. For concepts and symbols, please refer to [1-5].

Let P be a partially ordered set and A be a directed set. Using the order relationship, two types of order convergence can be defined in P: o_1-convergence and o_2-convergence.

Definition 1: In a partially ordered set P, a net $f = \{f(\alpha) | \alpha \in A\}$ o_1- convergens to f_0, denoted as $o_1 - \lim_{\alpha \in A} f(\alpha) = f_0$, if for some $\alpha_0 \in A$ and $\alpha \geq \alpha_0$, there exist $p_\alpha \in P, q_\alpha \in P$ such that for any $\beta \geq \alpha$

$$p_\alpha \leq p_\beta \leq f(\beta) \leq q_\beta \leq q_\alpha \text{ and } f_0 = \vee_{\alpha \geq \alpha_0} \{p_\alpha\} = \wedge_{\alpha \geq \alpha_0} \{q_\alpha\}.$$

Definition 2: In a partially ordered set P, a net $f = \{f(\alpha) | \alpha \in A\}$ o_2- convergens to f_0, denoted as $o_2 - \lim_{\alpha \in A} f(\alpha) = f_0$, if let

$$P_f = \{p \in P | \exists \alpha_0 \in A, \alpha \geq \alpha_0 \Rightarrow p \leq f(\alpha)\},$$
$$Q_f = \{q \in P | \exists \alpha_0 \in A, \alpha \geq \alpha_0 \Rightarrow q \geq f(\alpha)\},$$

then

$$f_0 = \vee P_f = \wedge Q_f.$$

In a partially ordered set P we have

$$o_1 - \lim_{\alpha \in A} f(\alpha) = f_0 \Rightarrow o_2 - \lim_{\alpha \in A} f(\alpha) = f_0.$$

$o_1 -$ convergence is introduced by Birkhoff in [9], also seen in [10]

Definition 3: In a partially ordered set P, if $f_1 = \{f(\alpha(\beta)) | \beta \in B\}$ is any subnet of net $f = \{f(\alpha) | \alpha \in A\}$ then there exists a net $\{\beta(\gamma) | \gamma \in \Gamma\}$ of elements in B such that the net $f' = \{f(\alpha(\beta(\gamma))) | \gamma \in \Gamma\}$ $o_1 - (o_2 -)$ convergens to f_0, we call that f $o_1^* - (o_2^* -)$ convergens to f_0, denoted as $o_1^*(o_2^*) - \lim_{\alpha \in A} f(\alpha) = f_0$.

From each order convergence, we can define an order topology. Let $F \subseteq P$, $f = \{f(\alpha) | \alpha \in A\}$ be a net in F, , if $o_1(o_2) - \lim_{\alpha \in A} f(\alpha) = f_0$, $f_0 \in F$, then F is said to be $o_1(o_2)-$ closed. In [6], Rennie pointed out that the $o_1 -$ topology and $o_2 -$ topology of a lattice are topologically equivalent, but in general partially ordered sets, they may not necessarily be equivalent, and strong convergence gives finer topologies. From $o_1^*(o_2^*) -$ convergence we can also lead to a topological structure, but $o_2^* -$ topology is equivalent to $o_2 -$ topology. Because the subnets $\{f(\alpha(\beta(\gamma))) | \gamma \in \Gamma\}$ of a net $f = \{f(\alpha) | \alpha \in A\}$ are still a net, $o_2^* -$ closeness can be derived from $o_2 -$ closeness. And $o_2 - \lim_{\alpha \in A} f(\alpha) = f_0 \Rightarrow o_2^* - \lim_{\alpha \in A} f(\alpha) = f_0$, therefore we can derive $o_2 -$ closeness from $o_2^* -$ closeness. If a partially ordered set P satisfies the condition of Theorem 2 in [7], then both $o_1 -$ convergence and $o_1^* -$ convergence lead to $o_1 -$ topology. The convergences derived from $o_1 -$ topology and $o_2 -$ topology are denoted as t_1 and t_2 respectively.

Theorem 1: In a partially ordered set P, $o_1^* -$ convergence is stronger than $t_1 -$ convergence, $o_2^* -$ convergence is stronger than $t_2 -$ convergence, i.e., for any net

$$f = \{f(\alpha) | \alpha \in A\},$$

$$o_1^* - \lim_{\alpha \in A} f(\alpha) = f_0 \Rightarrow t_1 - \lim_{\alpha \in A} f(\alpha) = f_0,$$

$$o_2^* - \lim_{\alpha \in A} f(\alpha) = f_0 \Rightarrow t_2 - \lim_{\alpha \in A} f(\alpha) = f_0.$$

Proof: Only provide proof for o_1^*, and the proof for o_2^* is similar. Suppose that $o_1^* - \lim_{\alpha \in A} f(\alpha) = f_0$, but $f(\alpha)$ does not $t_1 -$ converge to f_0. Then there exists $U \in C(f_0)$ where C (f_0) is the basic neighborhood system composed of open sets containin f_0, such that for every $\alpha \in A$ there exists $\beta \in A, \beta \geq \alpha, f(\beta) \notin U$. Let $B = \{\beta \in A | f(\beta) \notin U\}$. Then, there exists a co-tailed subnet $\{f(\alpha(\beta)) | \beta \in B\}$ of $f = \{f(\alpha) | \alpha \in A\}$, $f(\alpha(\beta)) \in P$ \U. By the definition of $o_1^* -$ convergence, there exists a subnet $\{f(\alpha(\beta(\gamma))) | \gamma \in \Gamma\}$ of $\{f(\alpha(\beta)) | \beta \in B\}$ such that $o_1 - \lim_{\gamma \in \Gamma} f(\alpha(\beta(\gamma))) = f_0$, $\{f(\alpha(\beta(\gamma))) | \gamma \in \Gamma\} \subseteq P$ \U. Since P \U is $o_1 -$ closed, $f_0 \in P$ \U , contradiction. □

From the above theorem, we can see that the Moore-Smith convergence of topology derived from order convergence in a partially ordered set is generally weaker than this type of order convergence.

Birkhoff introduced interval topology in [9] for general partially ordered sets, which means taking all the intervals $\{x | a \leq x \leq b\}, \{x | x \leq a\}$ and $\{x | x \geq b\}$ as closed set subbases and defining closed sets as any intersection of finite union of closed intervals. The interval topology is denoted as i.

In a partially ordered set, an open interval topology can also be defined, denoted as oi, which means taking all the interva $\{x | a \langle x \langle b\}, \{x | x \langle a\}$ and $\{x | x \rangle a\}$ as open set bases. It is easy to see from De Morgan formula that the interval topology and the open interval topology are equivalent in chains with antisymmetry [10]. But generally speaking, they are not equivalent.

Theorem 2: The interval topology is coarser than the open interval topology in a partially ordered set P.

Proof: Firstly, analyze the construction of interval topological open sets for a partially ordered set P. Let A be an interval topological open set. Then $A' = \bigcap_{a \in A} \bigcup_{n_a \in N_a} I_{n_a}$, where I_{n_a} is a closed interval as $\{x | a \leq x \leq b\}, \{x | x \leq a\}$ and $\{x | x \geq b\}$. From De Morgan formula we have

$$A = (\bigcap_{a \in A} \bigcup_{n_a \in N_a} I_{n_a})' = \bigcup_{a \in A} \bigcap_{n_a \in N_a} I'_{n_a}.$$

In general, $\{x | a \leq x \leq b\}' \supseteq \{x | x \langle a\} \cup \{x | x \rangle b\}$, $\{x | x \leq b\}' \supseteq \{x | x \rangle b\}$, $\{x | x \geq a\}' \supseteq \{x | x \langle a\}$. In the interval topology, take all sets with a shape of I'_{n_a} as the local subbasis of points. In open interval topology, take open interval $\{x | a \langle x \langle b\}, \{x | x \langle a\}$ and $\{x | x \rangle a\}$ as the local subbasis of points. If $F \subset P$ is a i closed set, it can be proven that any limit point f_0 of F in the open interval topology belongs to F. In fact, each set in the interval topological local subbasis of f_0 contains a set in the open interval topological local subbasis. If f_0 is a limit point of F in the open interval topology, then evevry set in open interval topology local subbasis of f_0 contains elements in F different from f_0, hence evevry set in interval topology local subbasis of f_0 contains elements in F different from f_0 and f_0 is a limit point of F in the interval topology, $f_0 \in F$. □

Frink proved in [10] that the o_1 topology of a lattice is finer than the interval topology. Below we will prove that topology o_2 is finer than the interval topology in a general partially ordered set. Because o_1 topology is finer than o_2 topology , it is also find that o_1 topology is finer than the interval topology.

Theorem 3: The o_2 topology of a partially ordered set P is finer than the open interval topology.

Proof: We need to prove that all closed intervals are o_2 closed. For example, $[a,b]=\{x \mid a \le x \le b\}$. Suppose that $\{f(\alpha) \mid \alpha \in A\} \subset [a,b], o_2-\lim_{\alpha \in A} f(\alpha) = f_0, f_0 = \vee M = \wedge N.$ Then, for every $m \in M$, $n \in N$ there exists $\alpha_0 \in A$ such that $m \le f(\alpha) \le n$ when $\alpha \ge \alpha_0$. Hence $b \in M*, a \in N^+$, and so $b = \vee M, a = \wedge N$, i.e., $f_0 \in [a,b]$. □

Theorem 4: The order topology is coarser than the open interval topology in a lattice.

Proof: Let L be a lattice. We prove that any o_2-closed subset is open interval topological closed. Suppose that $F \subseteq L$ is o_2 topological closed. All limit points of F in the open interval topology will belong to F. Take open intervals to be *oi* topological subbase of points. Suppose that f_0 is an limit points of F in the open interval topology. Then any open interval (a_α, b_α) and so closed interval $[a_\alpha, b_\alpha]$ contains a point $f(\alpha)$ different from f_0 at least. Cosider all finite closed intervals containing f_0. They form an directed set based on inclusion relationships, because for any two closed intervals $[a_\alpha, b_\alpha]$ and $[a_\beta, b_\beta]$, we have $[a_\alpha, b_\alpha] \cap [a_\beta, b_\beta] \quad = [a_\gamma, b_\lambda]$, wher $a_\gamma = a_\alpha \vee a_\beta, b_\gamma = a_\alpha \wedge b_\beta$. Hence we have a net $f(\overline{\alpha}), \overline{\alpha} \in \{[a_\alpha, b_\alpha]\}$ of elements in F. Let the set of the lower endpoints of all closed intervals containing f_0 be M, the set of the up endpoints of all closed intervals containing f_0 be N. Then we have $[a_\alpha, b_\alpha]$ for any $a_\alpha \in M$, and $[a_\beta, b_\beta]$ for any $b_\beta \in N$. Let $[a_\alpha, b_\alpha] \cap [a_\beta, b_\beta] = [a_\gamma, b_\gamma]$. Then, if $\overline{\gamma'} \ge \overline{\gamma}$, $f(\overline{\gamma'}) \in [a_{\gamma'}, b_{\gamma'}] \subseteq [a_\gamma, b_\gamma]$, and so $a_\alpha \le f(\overline{\gamma'}) \le b_\beta$. $a_\alpha \le f_0$ for any $a_\alpha \in M$. And if $f' \ge a_\alpha$ for all a_α, then $f' \ge f_0$. In fact, if there exists a element in L which greater than f_0, then f_0 is itself the lower endpoint of a closed intervals cotaining f_0. If there is no element greater than f_0 in L, then f' is greater than the lower end of all closed intervals containing

f_0, $f' \neq f_0$, then $f' \vee f_0 = f_0$, i.e., $f' \langle f_0$, hence $(f', f_o]$ contains aelement c of F and $f' < c$, a contradiction. Therefore,we have $f' = f_0$. Dually, we have $f_0 = \wedge b_\alpha$. Hence, $o_2 - \lim f(\overline{\alpha}) = f_0$. Since F is $o_2 -$closed, $f_0 \in F$. □

Because a chain is a lattice, and interval topology and open interval topology are equivalent in chains with antisymmetry, it is also known from Theorem 4 that order topology and interval topology are equivalent in chains with antisymmetry (see [10] p.571). Corollary 2 of Theorem 2 in [10] state that, if the i topology of a lattice L is T_2-type, then the i topology and ordered topology are equivalent, and the $o_2 -$convergence and the ordered topological convergence are equivalent. Below, we also provide a sufficient condition for the equivalence between order convergence and order topological convergence.

Theorem 5: In a complete chain with antisymmetry, i topological convergence, order convergence, and order topological convergence are equivalent.

Proof: Due to the existence of $\overline{\overline{\lim}}\, f$ and $\underline{\underline{\lim}}\, f$ for any net f in a complete chain, $o_1 -$convergence and $o_2 -$convergence are equivalent, and in chains with antisymmetry the order topology and i topology are equivalent. Therefore, it is only necessary to prove that the i topological convergence is equivalent to the $o_1 -$convergence. From Theorem 1 and Theorem 3, it can be seen that the $o_2 -$convergence is stronger than the i topological convergence. Now we prove that $i - \lim\limits_{\alpha \in A} f(\alpha) = f_0 \Rightarrow o_1 - \lim\limits_{\alpha \in A} f(\alpha) = f_0$ in complete chains with antisymmetry.We have $\underline{\underline{\lim}}\, f(\alpha) = \vee\{\wedge f(\alpha') \mid \alpha' \geq \alpha\} = f_0$. In fact, $n(\alpha) = \wedge\{f(\alpha') \mid \alpha' \geq \alpha\} \leq f_0$. Otherwisw, if for a certain $\beta, n(\beta) \rangle f_0$, then $f(\alpha) \geq n(\beta) \rangle f_0$ for $\alpha \geq \beta$. An open interval containing f_0 with the upper endpoint $n(\beta)$ cannot contain $f(\alpha)$ finally, which is contradictory. For $\vee\{\wedge\ f(\alpha') \mid \alpha' \geq \alpha\} \leq f_0$, we can prove that $\langle$ cannot be established. If $\vee\{\wedge\ f(\alpha') \mid \alpha' \geq \alpha\} = f' \langle f_0$, then an open interval containing f_0 with the lower endpoint f' cannot contain $f(\alpha)$ finally, and not all $\rangle f_0$ (otherwise, there exists some $n(\beta) = \wedge\{f(\alpha) \mid \alpha \geq \beta\} \geq f_0 \rangle$ $\rangle f'$, contradiction). Hence, there exists $f(\overline{\alpha})$ such that $f' \rangle f(\overline{\alpha}) \langle f_0$, and an open interval containing f_0 with the lower endpoint $f(\overline{\alpha})$ contains $f(\alpha)$ finally, i.e., there exists β, $f(\alpha) \rangle f(\overline{\alpha})$ when $\alpha \geq \beta$. Thus $n(\beta) = \wedge\{f(\alpha') \mid \alpha' \geq \beta\} \geq f(\overline{\alpha}) \rangle f'$, but f' is the supremum of all $n(\beta)$, contradiction. Therefore, $\vee\{\wedge f(\alpha') \mid \alpha' \rangle \alpha\} = f_0$. Dually, $\wedge\{\vee f(\alpha') \mid \alpha' \rangle \alpha\} = f_0$, hence $\overline{\overline{\lim}}\, f = \underline{\underline{\lim}}\, f$, and so $o_1 - \lim\limits_{\alpha \in A} f = f_0$. □

Rennie defined a lattice topology (L topology) for a lattice in [6]. The L topology can be similarly defined for general partially ordered sets. The L topology of a chain is i topology. From Theorem 4 of [5], the i topology of a lattice is generally coarser than the L topology. Mcshane defined a Dedekind topology for a partially ordered set in [11], denoted by D. If a set is o_1 topological closed, it is definitely Dedeking closed. In fact, let $S \subseteq P$ and K be a right directed subset of S. If there exists $\vee K$ in P, then $o_1 - \lim K = \vee K$, it follows from o_1- closness of S that $\vee K \in S$. Dually, if K' be a left directed subset of S. If there exists $\wedge K'$ in P, then $\wedge K \in S$. Hence S is Dedekingd closed. Therefore, o_1 topology is thicker than D topology. Frink defined an idear topology of a partial ordered set in [12], denoted by id. Ward proved that idear topological convergence is stronger that i topological convergence by using filter, i.e., the idear topology is finer that i topology. He also pointed that, if a partially ordered set P is T_2 type in i topology and is a compact space in idear topology, then i topology and idear topology are equavalent. In this case P is a complete lattice if P is a lattice, and idear topology and o topology are equivenlent. From Lemma 2 in [13] we see that idear topology and i topology are equavalent. But , in general idear topology is not thicker than o_2 topology. There are examples to indicate that idear topology and o topology are incomparable sometimes. Based on the above, we have

Theorem 6: In a partially ordered set P various intrinsic topologies have the following relationships:

$$oi \le i \ge o_2 \ge o_1 \ge D,$$
$$id \le i \ge o_2 \ge o_1 \ge D,$$

and in a lattice

$$L \le i \ge o \ge D$$
$$id \le i,\ \ oi \le o.$$

EQUIVALENCE OF INTRINSIC TOPOLOGY

According to Lemma 5 in [14], if each net of a partially ordered set P that convergens to f_0 in i topology always contains a monotonic one-sided subnet (i.e., always $\le f_0$ or $\ge f_0$), then i topology is equivalen to D totopology. We have obtained some result regarding the condition of equivalence between i topology and D topology (the order compatible uniqueness problem). For example, Work proved that a partially ordered set with finite divergence has a unique order compatible topology in [14]. Naito pointed that, If the i topology of lattice L satisfies the first countable axiom, then L has a unique order compatible topology. If a subnet $\{x_i \mid i = 1,2,3,\ldots..\}$ of a partially ordered set P satisfies the condition $x_1 \parallel x_2 \parallel \ldots . x_k \parallel x_{k+1} \parallel \ldots .$, and $i \ne j \Rightarrow x_i \ne x_j$, then $\{x_i\}$ is called a sacttered chain. If there exists a maximum number of elements for all scattered chainin in P, then P is said to be a finite scattered chain.

Lemma 1: In a partially ordered set P of finite scattered chain, if x_0 is the supremum of each subnet of net $\{x_\alpha \mid \alpha \in A\}$, then there exists a chain $C \subseteq \{x_\alpha \mid \alpha \in A\}$ such that $x_0 = \vee C$.

Proof: To refute it. Let the maximun number of scattered chain elements in P be k and $S = \{x_\alpha \mid \alpha \in A\}$. According to the Kuratowski lemma, let C_1 be a maximal chain in S, then $x_0 \ne \vee C_1$ by assumption. Therefore, it is impossible for a subnet of $\{x_\alpha \mid \alpha \in A\}$ to be included in

C_1, so there exists $\alpha_1 \in A$ such that $E_x(\alpha_1) \subseteq S \setminus C_1$. Let $E_x(\alpha_1)$ to be included in C_2, so there exists $\alpha_2 \in A$ such that $E_x(\alpha_2) \subseteq E_x(\alpha_1) \setminus C_2$. Similarly, to continue, we can obtain chains $C_1, C_2, \ldots, C_k, \ldots$ and $E_x(\alpha_1), E_x(\alpha_2), \ldots, E_x(\alpha_k), \ldots$, such that C_i is the maximum chain in $E_x(\alpha_{i-1})$, and $E_x(\alpha_i) \subseteq E_x(\alpha_{i-1}) \setminus C_i (i = 1,2,\ldots,k,\ldots, E_x(\alpha_0) = S)$. Now we take $x_{k+1} \in E_x(\alpha_{k-1}) \setminus C_k$, then there exists $x_k \in C_k$ such that $x_{k+1} \parallel x_k$. Since $x_k \in E_x(\alpha_{k-2}) \setminus C_{k-1}$, there exists $x_{k-1} \in C_{k-1}, x_k \parallel x_{k-1}$. Continuing to work will yield $x_i \in C_i (i = 1,2,\ldots,k,\ldots), i_1 \neq i_2 \Rightarrow x_{i_1} \neq x_{i_2}$, and $x_{k+1} \parallel x_k \parallel x_{k-1} \parallel \ldots \parallel x_2 \parallel x_1$, contradiction. Therefore, there exists a chain $C \subseteq S$ such that $x_0 = \vee C$. □

Lemma 2: In a partially ordered set P with finite divergence, if a set is o_1- convergence closed with respect to transfinite sequence, then it is o_1 topological closed.

Proof: Suppose that a subset E of P is o_1-convergence closed with respect to transfinite sequence. If a net $\{x_\alpha \mid \alpha \in A\}$ of elements in E is o_1-convergent, then $o_1^* - \lim_{\alpha \in A} x_\alpha = x_0$. From Theorem 1, $t_1 - \lim_{\alpha \in A} x_\alpha = x_0$, and $i - \lim_{\alpha \in A} x_\alpha = x_0$ by Theorem 6. Because a partially ordered set of finite scattered chain is finite divergence, according to Lemma 4 in [14] there exists a subnet $\{x_{\alpha_\beta} \mid \beta \in B\}$ of $\{x_\alpha \mid \alpha \in A\}$ such that $x_{\alpha_\beta} \leq x_0$ or $x_{\alpha_\beta} \geq x_0$ for all $\beta \in B$. Furthermore, since each subnet of the t-topological convergent net is i-topological convergent, from Lemma 5 in [14] x_0 is the supremun of every subnet $\{x_{\alpha_\beta} \mid \beta \in B\}$. From Lemma 1 there exists a chain C of $\{x_{\alpha_\beta} \mid \beta \in B\}$ such that $x_0 = \vee C$. Let $C = \{x_\delta \mid \delta \in \Lambda\}$ where $x_{\delta_1} \leq x_{\delta_2}$ if and only if $\delta_1 \leq \delta_2$. Hence $o_1 - \lim_{\delta \in \Lambda} x_\delta = x_0$. By the assumption, $x_0 \in E$, E is o_1 closed. □

From proof of the above lemma it is not difficult to see that, in a partially ordered set with finite divergence, if a set is o_2-convergence closed with respect to transfinite sequence, then it is also o_2-closed.

Lemma 3: In a partially ordered set P with finite divergence, $i \geq L$ if and only if the limit of each $L-$topology convergent net in P is its midpoint.

Proof: **Necessity.** If $i \geq L, L - \lim_{\alpha \in A} x_\alpha = x_0$, then $i - \lim_{\alpha \in A} x_\alpha = x_0$. Let Ψ' be the filter base connected to $\{x_\alpha \mid \alpha \in A\}$, and $\Psi = \overline{\Psi'}$ be the filter generated by Ψ'. From [15] we see that $i - \lim \Psi = x_0$. It follows from Lemma 1 in [10] that x_0 is a midpoint of Ψ and it also a midpoint of $\{x_\alpha \mid \alpha \in A\}$.

Sufficiency: Suppose that limit of each $L-$topological convergence net in P is its midpoint, we can prove that the ordered structure of P is semi continuous with respect to the $L-$topology. (The definition of semi continuity can be found in [16]). Let $L - \lim_{\alpha \in A} x_\alpha = x_0$, and

$x_\alpha \geq a$ for all $\alpha \in A$. Since $x_0 \in M_x$, it follows from Lemma 4 and Lemma 5 in [14] that there exists a subnet $\{x_{\alpha_\beta} \mid \beta \in B\} \subseteq \{x_\alpha \mid \alpha \in A\}$ such that $x_0 = \vee\{x_{\alpha_\beta} \mid \beta \in B\}$ (if $x_0 = \wedge\{x_{\alpha_\beta} \mid \beta \in B\}$, the proof is similary). For any subnet $\{x_{\alpha_{\beta_\gamma}} \mid \gamma \in \Gamma\}$ of $\{x_{\alpha_\beta} \mid \beta \in B\}$, $x_0 \in [\{x_{\alpha_{\beta_\gamma}} \mid \gamma \in \Gamma\}]^*$ because of $L - \lim_{\gamma \in \Gamma} x_{\alpha_{\beta_\gamma}} = x_0$. Hence $x_0 = \vee\{x_{\alpha_{\beta_\gamma}} \mid \gamma \in \Gamma\}$. From Lemma 6 in [14] , there exists a updirected subset $M \subseteq \{x_{\alpha_\beta} \mid \beta \in B\}$ such that $x_0 = \vee M$. Let $M = \{x_\delta \mid \delta \in \Lambda\}$ where $x_{\delta_1} \leq x_{\delta_2}$ if and only if $\delta_1 \leq \delta_2$. Then $o_1 - \lim_{\delta \in \Lambda} x_\delta = x_0$, and so $i - \lim_{\delta \in \Lambda} x_\delta = x_0$ by Theorem 6. Since $x_\delta \geq a$ and the ordered structure of P is semi continuous with respect to interval topology, $x_0 \geq a$. This proves that the ordered structure of P is semi continuous with respect to L topology. The interval topology is the coarsest topology that makes the ordered structure of P is semi continuous ([16]), so $i \geq L$. □

Further, we have

Theorem 7: In a partially ordered set P with finite divergence, $i = L$ if and only if the limit of each L-topological convergemt net in P is midpoint.

Proof: Ncessity is clear. Now we prove sufficiency. Because a partially ordered set of finite scattered chain is finite divergence, so $i \geq L$ by Lemma 3. Now we prove $i \leq L$. Let $E \subseteq P$ be L topological closed, and $\{x_\alpha \mid \alpha \in A\} \subseteq E$, $i - \lim_{\alpha \in A} x_\alpha = x_0$. It can be assumed that $x_\alpha \neq x_0$. From Lemma 4 in [14], there exists $\alpha_1 \in A$ such that every element of the truncated subnet $E_x(\alpha_1)$ of $\{x_\alpha \mid \alpha \in A\}$ canbe comparable to x_0. This trucated subnet is taken as $\{x_\alpha \mid \alpha \in A\}$ itself. Hence there exists a subnet $\{x_{\alpha_\beta} \mid \beta \in B\}$ of $\{x_\alpha \mid \alpha \in A\}$ such that $x_{\alpha_\beta} \leq x_0$ (the case of $x_{\alpha_\beta} \geq x_0$ is similar), and so $i - \lim_{\beta \in B} x_{\alpha_\beta} = x_0$ and x_0 is amidpoint of $\{x_{\alpha_\beta} \mid \beta \in B\}$. From Lemma 5 in [14], $x_0 = \vee\{x_{\alpha_\beta} \mid \beta \in B\}$. Since every subnet of $\{x_{\alpha_\beta} \mid \beta \in B\}$ is i topological convergent, x_0 is the supermum of every subnet of $\{x_{\alpha_\beta} \mid \beta \in B\}$. From Lemma 1, there existsa chain $C = \{x_\delta \mid \delta \in \Lambda\}$ ($\delta_1 \leq \delta_2$ if and only if $x_{\delta_1} \leq x_{\chi_2}$) such that $x_0 = \vee C$. Therefore, $o_1 - \lim_{\delta \in \Lambda} x_\delta = x_0$. From the L closeness of E, $x_0 \in E$. □

In a partially ordered set P, if the number of elements that are incomparable to every element is finite, then P is said to be locally finite divergence.

Theorem 8: In a locally finite divergence partially ordered set P, $i = oi$.

Proof: From Theorem 2 we ned to prove $i \leq oi$. Now we analyze the structure of oi topological closed set. If $F \subseteq P$ is oi closed, then $P \setminus F = \bigcup_{\alpha \in A} \bigcap_{n_\alpha \in N_\alpha} I_{n_\alpha}$, where I_{n_α} is a open interval $[\leftarrow, x)$ or $(x, \rightarrow]$. Then $F = \bigcap_{\alpha \in A} \bigcup_{n_\alpha \in N_\alpha} (P \setminus I_{n_\alpha})$, where $P \setminus I_{n_\alpha}$ is a subnet shaped like $(x, \rightarrow] \cup \{y \mid y \parallel x\}$ or

$[\leftarrow, x) \cup \{y \mid y \parallel x\}$. Since $\{y \mid y \parallel x\}$ is finite, these sets are i topological closed. Therefore F is i topological closed. □

Finally, let us discuss the relationship between order convergence and open interval topological convergence.

Theorem 9: In a conditionally complete partially ordered set P, for any net $\{x_\alpha \mid \alpha \in A\}$, $oi - \lim_{\alpha \in A} x_\alpha = x_0 \Rightarrow o_1 - \lim_{\alpha \in A} x_\alpha = x_0$.

Proof: Suppose that $oi - \lim_{\alpha \in A} x_\alpha = x_0$ and Υ_{x_0} is an oi topoligical neighborhood system of x_0 [17]. If $(a,b) \in \Upsilon_{x_0}$, then there exists $\alpha_0 \in A$ such that $x_\alpha \in (a,b)$ when $\alpha \geq \alpha_0$. Let $p_\alpha = \wedge E_x(\alpha)$ and $q_\alpha = \vee E_x(\alpha)$ when $\alpha \geq \alpha_0$. If $\alpha \leq \beta$, then $p_\alpha \leq p_\beta \leq x_\beta \leq q_\beta \leq q_\alpha$. From Theorem 2, $i - \lim_{\alpha \in A} x_\alpha = x_0$. It follows from 2.2 in [15] that the filter generated by the filter base connected by $\{x_\alpha \mid \alpha \in A\}$ is interval topological convergent. From Lemma 1 in [7], x_0 is a midpoint of $\{x_\alpha \mid \alpha \in A\}$. Hence $p_\alpha \leq x_0 \leq q_\alpha$. Let $x_1 = \vee_{\alpha \geq \alpha_0} \{p_\alpha\}, x_2 = \wedge_{\alpha \geq \alpha_0} \{q_\alpha\}$, then $x_1 \leq x_0 \leq x_2$ and there exists $\alpha' \in A$ such that $x_\alpha \in (x_1, x_2)$ when $\alpha \geq \alpha'$. At this point, $p_\alpha = \wedge E_x(\alpha) \geq x_1$. Hence $x_1 = x_0$. Similarly, $x_2 = x_0$. Therefore, $o_1 - \lim_{\alpha \in A} x_\alpha = x_0$. □

Declarations

Ethical Compliance: All procedures performed in studies involving human participants were in accordance with the ethical standards of the institutional and/or national research committee and with the 1964 Helsinki Declaration and its later amendments or comparable ethical standards.

Consent to participate and consent to publish. We have no conflict of interest with anyone. We have no funding.

Statement

The datasets generated and analysed during the current study are available .

References

[1] Daoron Ton, The Intransic Topologies of an Lattice Group, Journal of China University of Science and Technology,1982(supplement),1-9.

[2] Daoron Ton, A Construction Theorem of a Complete l − group, Journal of Shangdong Normal University, No.2, 1985, 6-13.

[3] Daoron Ton, Topological Completion of a Commutative l − group, Acta Mathematica Sinica, 1986,29(2), 249-252.

[4] Daoron Ton, Complete Distributivity of an l − group, Chin.Ann.of Math.,9B(1), 1988,107-111.

[5] Daoron Ton, The Order Convergences in a Partially Ordered Set, Journal of Shangdong Normal UNiversity, No.3, 1986, 1-6.

[6] B.C.Rennie, Lattice, Proc. London. Math. Soc.,1951,52;386-400.

[7] A.J.Ward, On Relations beyween Certain Intransic Topologies in Partially Ordered Sets, Proc.Cambridge PHilos. So., 1955,51;254-261.

[8] J.L.Kelley, General Topology, New York Van Nostrad,1964.

[9] G.Birkhoff, Lattice Theory, 3rd ed., Providence Amer. Math. Soc. Colloquium XX V 1967.

[10] O.Frink, Topology in Lattice, Trans. Amer, Math. Soc., 1942;51;569-582.

[11] E.J.Macshane, Orde-preseving Maps and Integration Processes, Annals of Mathematics Adudies (Num.31), Princeton University Press,1953.

[12] O.Frink, Ideal in Partially Ordered Sets, Amer. Math. Mon., 1954, 61,223-234.

[13] E.S.Work, On Order-Covergence, Proc. Amer. Math. Soc., 1961,379-384.

[14] E.S.Work, Order-Compatible Topologies on a Poset, Proc. Amer. Math. Soc., 1958;9;524-529.

[15] R.G.Bartle, Net and Filter in Topology, Amer. Math. Month. 1955; 62; 551-557.

[16] L .E.Ward, Partially Ordered Topological Space, Proc. Amer. Math. Soc., 1954; 5;144-161.

[17] H.J.Kowalsky, Topological Space, New York, Academic Press,1955.

[18] A.M.W.Class, Partially Oderered Groups, World Scientific,1999.

[19] Maoting Tong, Daorong Ton, A Local Strong Solution of the Navier-Stokes Problem in L2(Ω), arXiv, *30 Dec 2020.*

[20] Maoting Tong, Strong Maximal $L_q - L_p -$ Estimates for the Solution of the Stokes Equation, *Transactions on Engineering and Computing Sciences*, *11*(1)（2023）, 132–142. https://doi.org/10.14738/tecs.111.13785.