European Journal of Applied Sciences – Vol. 13, No. 05
Publication Date: September 07, 2025
DOI:10.14738/aivp.1305.19327.
Ayala, Y. S. S. & López, D. S. (2025). Existence and Continuous Dependence of the Local Solution of a Non-Homogeneous Equation of Order Multiple of Four. *European Journal of Applied Sciences, Vol - 13(05).* 48-58.

Existence and Continuous Dependence of the Local Solution of a Non-Homogeneous Equation of Order Multiple of Four

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ABSTRACT

In this article, we prove that initial value problem associated to the non-homogeneous n-th order equation in periodic Sobolev spaces with n multiple of four has a local solution in [0, T] with T > 0, and the solution has continuous dependence with respect to the initial data and the non-homogeneous part of the problem. We do this in a intuitive way using Fourier theory and introducing a C_o- Semigroup inspired by the work of Iorio [1] and Santiago [5].

Keywords: Uniqueness solution, fourth order equation, non-homogeneous equation, periodic Sobolev spaces, Fourier Theory, calculus in Banach spaces.

INTRODUCTION

First, we want to comment that from Theorem 3.1 in [6], we have that the homogeneous problem is globally well posed and, in addition to the equality (3.2) in [6], we have the continuous dependence of the solution of homogeneous problem respect to the initial data.

In this work, in Theorem 3.2 we will prove the existence and uniqueness of the local solution for the non-homogeneous problem and from inequality (3.8) we will get the continuous dependence of the solution with respect to the initial data and respect to the non-homogeneous part.

Thus, in both homogeneous and non-homogeneous cases, the estimatives are made from the explicit form of the solution, that is, by applying the Fourier transform to the respective equation. We cite some works about n-th order equation [1], [6] and for dissipative properties of systems [2].

Our article is organized as follows. In section 2, we indicate the methodology used and cite the references used. In section 3, we proved the main results for the non-homogeneous even order equation. Finally, in section 4, we give the conclusions of our study.

METHODOLOGY

As a theoretical framework in this article, we use the existence and regularity results of [6]. Also, we use the references [1], [4], [5], [6], [7] and [3] for the Fourier theory in periodic Sobolev spaces, and differential and integral calculus in Banach spaces.

MAIN RESULTS

Using the Fourier transform, we will prove that the non-homogeneous problem has a unique solution and it continuously depends respect to the initial data and the non homogeneity in compact intervals. Remember that the homogeneous problem (F=0) is dissipative; see [6].

The Non-homogeneous Problem (P_n^F) is Locally Well Posed

Theorem 3.1 Let s a fixed real number, n an even number multiple of four, $F \in C\big([0,T], H^s_{per}\big)$, where T>0, $\{S(t)\}_{t\in R}$ the semigroup of class C_o of homogeneous case (F = 0), introduced in the Theorem 3.2 from [6], and

$$u_p(t) := \int_0^t S(t-\tau)F(\tau)d\tau.$$

Then $u_p \in C\big([0,T], H^s_{per}\big) \cap C^1\big([0,T], H^{s-n}_{per}\big)$ and satisfies

$$\left| \begin{array}{l} \partial_t u_p + \partial_x^n u_p = F(t) \ \in \ H^{s-n}_{per} \\ u_p(0) = 0 \end{array} \right. \tag{3.1}$$

with the derivative given by

$$\lim_{h\to 0} \left\| \frac{u_p(t+h)-u_p(t)}{h} + \partial_x^n u_p - F(t) \right\|_{s-n} = 0. \tag{3.2}$$

Proof: We remark that $S(t-\tau)F(\tau) \in H^s_{per}$, $\forall \tau \in (0,t)$ and $\tau \to S(t-\tau)F(\tau)$ is continuous in [0, t] then exists $\underbrace{\int_0^t S(t-\tau)F(\tau)d\tau}_{u_p(t)=} \in H^s_{per}$. Now, we will prove $u_p \in C\big([0,T], H^s_{per}\big)$, that is, $\| \, u_p(t+h) - u_p(t) \, \|_s \to 0$ when $h \to 0$.

Let $h > 0$

$$\begin{aligned} u_p(t+h) - u_p(t) \quad &= \quad \int_0^{t+h} S(t+h-\tau)F(\tau)d\tau - \int_0^t S(t-\tau)F(\tau)d\tau \\ &= \quad \int_0^t \{S(t+h-\tau) - S(t-\tau)\}F(\tau)d\tau \\ &\quad\ + \int_t^{t+h} S(t+h-\tau)F(\tau)d\tau \end{aligned}$$

taking the norm $\|\cdot\|_s$ we obtain

$$\left\|u_p(t+h)-u_p(t)\right\|_s \leq \underbrace{\left\|\int_0^t \{S(t+h-\tau)-S(t-\tau)\}F(\tau)d\tau\right\|_s}_{I_1:=} + \underbrace{\left\|\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau\right\|_s}_{I_2:=}.$$

Using the M-Test of Weierstrass we get

$$\begin{aligned} I_1 &\leq \int_0^t \|S(t+h-\tau)F(\tau)-S(t-\tau)F(\tau)\|_s \, d\tau \\ &\leq \frac{\epsilon}{2T}\cdot\int_0^t d\tau = \frac{\epsilon t}{2T} \leq \frac{\epsilon T}{2T} = \frac{\epsilon}{2} \end{aligned}$$

since ∃δ > 0 such that:

$$\text{if } |h|<\delta \text{ then } \|S(t+h-\tau)F(\tau)-S(t-\tau)F(\tau)\|_s < \frac{\epsilon}{2T}, \ \forall\tau\in(0,t).$$

Using the mean value Theorem in H^s_{per} we obtain

$$\frac{1}{h}\cdot\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau \rightarrow S(0)F(t) = F(t)$$

when $h \to 0$. Then

$$\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau = \underbrace{h}_{\to 0}\cdot\underbrace{\frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau}_{\to F(t)} \rightarrow 0$$

when $h \to 0$. So, we have

$$I_2 = \left\|\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau\right\|_s \to 0$$

when $h \to 0$. That is $I_2 < \frac{\epsilon}{2}$ whenever $|h| < \delta^*$.

Therefore, $\left\|u_p(t+h)-u_p(t)\right\|_s \leq I_1+I_2<\epsilon$ whenever $|h|<\min\{\delta,\delta^*\}$.

From definition of $u_p(t)$ we have $u_p(0)=0$.

Now, we will prove that $\exists\, \partial_t u_p(t)$ in H^{s-n}_{per}. In effect

$$\frac{u_p(t+h)-u_p(t)}{h}$$
$$=\frac{1}{h}\left\{\int_0^{t+h} S(t+h-\tau)F(\tau)d\tau - \int_0^{t} S(t-\tau)F(\tau)d\tau\right\}$$
$$=\frac{1}{h}\left\{\int_0^{t} \{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)\}\,d\tau + \int_t^{t+h} S(t+h-\tau)F(\tau)d\tau\right\}$$
$$=\frac{1}{h}\int_0^{t} \{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)\}\,d\tau + \frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau.$$

Using the mean value Theorem in H^{s-n}_{per} , we obtain

$$\frac{1}{h}\cdot\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau \quad\rightarrow\quad S(0)F(t) \;=\; F(t) \tag{3.3}$$

when $h \rightarrow 0$.

Since

$$\frac{S(t+h-\tau)F(\tau)-S(t-\tau)F(\tau)}{h}$$

converges uniformly to $\partial_t\{S(t-\tau)F(\tau)\}$ in H^{s-n}_{per} $\forall\tau \in [0,t]$, we obtain that

$$\int_0^t \frac{S(t+h-\tau)F(\tau)-S(t-\tau)F(\tau)}{h}\,d\tau$$

converges to $\int_0^t \partial_t\{S(t-\tau)F(\tau)\}\,d\tau$ when $h \rightarrow 0$.

Now, we remark that $\partial_t\{S(t-\tau)F(\tau)\} = (-\partial_x^n)S(t-\tau)F(\tau)$ in H^{s-n}_{per} .

Since $(-\partial_x^n)$ is a closed operator, then

$$\int_0^t (-\partial_x^n)S(t-\tau)F(\tau)d\tau = (-\partial_x^n)\int_0^t S(t-\tau)F(\tau)\,d\tau.$$

Therefore,

$$\int_0^t \partial_t\{S(t-\tau)F(\tau)\}\,d\tau = (-\partial_x^n)\underbrace{\int_0^t S(t-\tau)F(\tau)\,d\tau.}_{u_p(t)=}$$

That is,

$$\int_0^t \frac{S(t+h-\tau)F(\tau)-S(t-\tau)F(\tau)}{h}\,d\tau \quad\rightarrow\quad (-\partial_x^n)u_p(t) \tag{3.4}$$

when $h \to 0$.

Finally, in H_{per}^{s-n} , using (3.3) and (3.4) we get

$$\underbrace{\exists \lim_{h\to 0}\frac{u_p(t+h)-u_p(t)}{h}}_{\partial_t u_p(t)=} = (-\partial_x^n)u_p(t) + F(t)\,.$$

That is, $\exists\ \partial_t u_p(t)$ and it also satisfies $\partial_t u_p(t) = (-\partial_x^n)u_p(t) + F(t)$.

Using that $\|\partial_x^m f\|_{s-m} \le \|f\|_s$, $\forall m \in Z^+$, $\forall\, f \in H_{per}^s$, we obtain

$$\begin{aligned}
&\left\|\partial_t u_p(t+h) - \partial_t u_p(t)\right\|_{s-n}\\
&= \left\|(-\partial_x^n)u_p(t+h) + F(t+h) - \{(-\partial_x^n)u_p(t) + F(t)\}\right\|_{s-n}\\
&\le \|F(t+h)-F(t)\|_{s-n} + \left\|(-\partial_x^n)\{\,u_p(t+h)-u_p(t)\}\right\|_{s-n}\\
&= \|F(t+h)-F(t)\|_{s-n} + \left\|\partial_x^n\{\,u_p(t+h)-u_p(t)\}\right\|_{s-n}\\
&\le \|F(t+h)-F(t)\|_{s-n} + \left\|\,u_p(t+h)-u_p(t)\right\|_s\,.
\end{aligned}$$

So, since $u_p(t) \in H_{per}^s \subset H_{per}^{s-1} \subset \cdots \subset H_{per}^{s-(n-1)} \subset H_{per}^{s-n}$, we have $F\colon [0,T] \to H_{per}^{s-n}$ is continuous and as $u_p\colon [0,T] \to H_{per}^s$ is continuous, then $\partial_t u_p\colon [0,T] \to H_{per}^{s-n}$ is continuous, that is, $\partial_t u_p \in C([0,T], H_{per}^{s-n})$.

Therefore, $u_p \in C([0,T], H_{per}^s) \cap C^1([0,T], H_{per}^{s-n})$.
■

Theorem 3.2 Let s a fixed real number, n an even number multiple of four, $\varphi \in H_{per}^s$, $F \in C\big([0,T], H_{per}^s\big)$, where T>0 and $\{S(t)\}_{t\ge 0}$ the semigroup of class C_o in H_{per}^s as in Theorem 3.1, then

The function

$$u^F(t) \coloneqq S(t)\varphi + \underbrace{\int_0^t S(t-\tau)F(\tau)\,d\tau}_{u_p(t)=}, \qquad t \in [0,T] \tag{3.5}$$

belongs to $C([0,T], H_{per}^s) \cap C^1([0,T], H_{per}^{s-n})$ and

$u^F(t)$ is the unique solution of

$$(P_n^F)\quad \left|\begin{array}{l} u_t + \partial_x^n u = F(t) \in H_{per}^{s-n}\\ u(0) = \varphi \end{array}\right. \tag{3.6}$$

with the derivative given by

$$\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}+\ \partial_x^n u-F(t)\right\|_{s-n}=0. \tag{3.7}$$

Let $\psi_j \in H^s_{per}$, $F_j \in C\big([0,T],H^s_{per}\big)$, $j=1,2$. The map $\psi \to u$ is continuous in the following sense. Let u_1 and u_2 the corresponding solutions to initial data ψ_1 and ψ_2, and with non homogeneity F_1 and F_2 respectively. Then

$$\begin{aligned}
\|u_1(t)-u_2(t)\|_s &\leq \|\psi_1-\psi_2\|_s + T\|F_1-F_2\|_{\infty,s},\ \ t\in[0,T], && (3.8)\\
\underbrace{sup_{t\in[0,T]}\|u_1(t)-u_2(t)\|_s}_{\|u_1-u_2\|_{\infty,s}=} &\leq \|\psi_1-\psi_2\|_s + T\|F_1-F_2\|_{\infty,s} && (3.9)
\end{aligned}$$

$$\begin{aligned}
\|\partial_t u_1(t)-\partial_t u_2(t)\|_{s-n} &\leq \|u_1(t)-u_2(t)\|_s+\ \|F_1-F_2\|_{\infty,s-n}\,, t\in[0,T],\\
&\leq \|u_1-u_2\|_{\infty,s}+\|F_1-F_2\|_{\infty,s}\\
&\leq \|\psi_1-\psi_2\|_{\infty,s}+(T+1)\|F_1-F_2\|_{\infty,s} && (3.10)
\end{aligned}$$

where we have used the notation

$$\|h\|_{\infty,r} \coloneqq\ sup_{t\in[0,T]}\|h(t)\|_r, \qquad h\ \in C\big([0,T],H^r_{per}\big). \tag{3.11}$$

Proof: Previously, we will proceed to obtain the candidate solution of (P_n^F). To achieve this, we apply the Fourier transform to the non-homogeneous equation (P_n^F)

$$\partial_t u(t)+\partial_x^n u(t)=F(t),$$

and obtain

$$\partial_t \hat{u}(k,t)+k^n\hat{u}(k,t)=\hat{F}(k,t),$$

that for each $k\in\mathbb{Z}$, it is a non-homogeneous ordinary differential equation with initial data $\hat{u}(k,0)=\hat{\varphi}(k)$.

Thus, we propose an uncoupled system of non-homogeneous first-order equations:

$$(\Omega_k)\left|\begin{array}{l}\hat{u}\in C([0,T],\ell^2_s(\mathbb{Z}))\\ \partial_t\hat{u}(k,t)+k^n\hat{u}(k,t)=\hat{F}(k,t)\\ \hat{u}(k,0)=\hat{\varphi}(k)\ \ with\ \hat{\varphi}\in\ell^2_s(\mathbb{Z})\end{array}\right.$$

for all $k\in\mathbb{Z}$, which we will solve below.

Let $k\ \neq\ 0$, multiplying by the integrating factor $e^{k^n t}$ to the differential equation of (Ω_k), we obtain

$$\partial_t\big\{e^{k^n t}\hat{u}(k,t)\big\}=e^{k^n t}\hat{F}(k,t),$$

integrating from 0 to t, we have

$$\int_0^t \partial_\tau\left\{e^{k^n\tau}\hat{u}(k,\tau)\right\}d\tau = \int_0^t e^{k^n\tau}\hat{F}(k,\tau)\, d\tau\,;$$

then,

$$e^{k^n t}\hat{u}(k,t) - \hat{u}(k,0) = \int_0^t e^{k^n\tau}\hat{F}(k,\tau)\, d\tau.$$

That is,

$$\begin{aligned}\hat{u}(k,t) &= e^{-k^n t}\hat{\varphi}(k) + e^{-k^n t}\int_0^t e^{k^n\tau}\hat{F}(k,\tau)\, d\tau \\ &= e^{-k^n t}\hat{\varphi}(k) + \int_0^t e^{-k^n(t-\tau)}\hat{F}(k,\tau)\, d\tau\,.\end{aligned}$$

If $k = 0$, the non-homogeneous ODE is

$$\left|\begin{array}{l}\partial_t\hat{u}(0,t) = \hat{F}(0,t) \\ \hat{u}(0,0) = \hat{\varphi}(0).\end{array}\right.$$

integrating from 0 to t, we obtain

$$\int_0^t \partial_\tau\hat{u}(0,\tau)d\tau = \int_0^t \hat{F}(0,\tau)\, d\tau\,;$$

then

$$\hat{u}(0,t) - \hat{u}(0,0) = \int_0^t \hat{F}(0,\tau)\, d\tau\,;$$

that is

$$\hat{u}(0,t) = \hat{\varphi}(0) + \int_0^t \hat{F}(0,\tau)\, d\tau\,;$$

Finally,

$$\hat{u}(k,t) = e^{-k^n t}\hat{\varphi}(k) + \int_0^t e^{-k^n(t-\tau)}\hat{F}(k,\tau)\, d\tau\ ,\forall k \in \mathbb{Z}\,.$$

The candidate to be the solution of (P_n^F) is

$$\begin{aligned}u(t) &= \sum_{k=-\infty}^{+\infty}\hat{u}(k,t)\,\varphi_k \\ &= \sum_{k=-\infty}^{+\infty}\left\{e^{-k^n t}\hat{\varphi}(k) + \int_0^t e^{-k^n(t-\tau)}\hat{F}(k,\tau)\, d\tau\right\}\varphi_k \\ &= \sum_{k=-\infty}^{+\infty} e^{-k^n t}\hat{\varphi}(k)\,\varphi_k + \sum_{k=-\infty}^{+\infty}\left\{\int_0^t e^{-k^n(t-\tau)}\hat{F}(k,\tau)\, d\tau\right\}\varphi_k\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=-\infty}^{+\infty} e^{-k^n t}\hat{\varphi}(k)\,\varphi_k + \int_0^t \sum_{k=-\infty}^{+\infty} e^{-k^n(t-\tau)}\hat{F}(k,\tau)\varphi_k\,d\tau \\
&= \underbrace{S(t)\varphi}_{u_h(t):=} + \underbrace{\int_0^t S(t-\tau)F(\tau)\,d\tau}_{u_p(t):=},
\end{aligned}
$$

where u_h is the solution to the homogeneous equation (P_n^0), which has already been proven and u_p is the particular solution to (P_n^F) with a null initial condition, which was also proved in the previous theorem.

We work the following steps.

Let $u(t) := u^F(t) = S(t)\varphi + u_p(t)$, we will prove that $u \in C\big([0,T], H_{per}^s\big) \cap C^1\big([0,T], H_{per}^{s-n}\big)$. In effect, as $S(\cdot)\varphi \in C\big([0,T], H_{per}^s\big)$ and $u_p(\cdot) \in C\big([0,T], H_{per}^s\big)$ then $u(\cdot) = S(\cdot)\varphi + u_p(\cdot) \in C\big([0,T], H_{per}^s\big)$. Moreover, as $S(\cdot)\varphi \in C^1\big([0,+\infty), H_{per}^{s-n}\big)$ and $u_p(\cdot) \in C^1\big([0,T], H_{per}^{s-n}\big)$ then $u(\cdot) = S(\cdot)\varphi + u_p(\cdot) \in C^1\big([0,T], H_{per}^{s-n}\big)$.

We will prove that u is the solution of (P_n^F). In effect, we know that $\exists \partial_t S(t)\varphi$ and $\exists \partial_t u_p(t)$ in H_{per}^{s-n}, then

$$
\begin{aligned}
\partial_t u(t) &= \partial_t \underbrace{S(t)\varphi}_{u_h(t):=} + \partial_t u_p(t) \\
&= -\partial_x^n u_h(t) - \partial_x^n u_p(t) + F(t) \\
&= -\partial_x^n\{u_h(t) + u_p(t)\} + F(t) \\
&= -\partial_x^n u(t) + F(t)
\end{aligned}
$$

in H_{per}^{s-n}, where $u_h(\cdot)$ is solution of the homogeneous equation ($P_n := P_n^0$). Also, $u(0) = u_h(0) + u_p(0) = \varphi + 0 = \varphi$.

Let $\psi_j \in H_{per}^s$ and $F_j \in C\big([0,T], H_{per}^s\big)$ for $j = 1,2$, then

$$
u_j(t) = S(t)\psi_j + \int_0^t S(t-\tau)F_j(\tau)d\tau
$$

is solution of $(P_n^{F_j})$ with initial data $u_j(0) = S(0)\psi_j = \psi_j$, for $j = 1,2$.

Then

$$
u_1(t) - u_2(t) = S(t)\{\psi_1 - \psi_2\} + \int_0^t S(t-\tau)\{F_1(\tau) - F_2(\tau)\}d\tau.
$$

From where we obtain, for t<T:

$$\begin{aligned}\|u_1(t)-u_2(t)\|_s &\leq \|S(t)\{\psi_1-\psi_2\}\|_s+\int_0^t \|S(t-\tau)\{F_1(\tau)-F_2(\tau)\}\|_s d\tau\\ &\leq \quad \|\psi_1-\psi_2\|_s+\int_0^t \|F_1(\tau)-F_2(\tau)\|_s d\tau\\ &\leq \quad \|\psi_1-\psi_2\|_s+sup_{\tau\in[0,T]}\|F_1(\tau)-F_2(\tau)\|_s\int_0^t d\tau\\ &\leq \quad \|\psi_1-\psi_2\|_s+T\cdot sup_{\tau\in[0,T]}\|F_1(\tau)-F_2(\tau)\|_s.\end{aligned}$$

Therefore,

$$sup_{\tau\in[0,T]}\|u_1(t)-u_2(t)\|_s \leq \|\psi_1-\psi_2\|_s+T\cdot sup_{\tau\in[0,T]}\|F_1(\tau)-F_2(\tau)\|_s.$$

On the other hand, in H_{per}^{s-n} we have

$$\begin{aligned}\partial_t u_j(t) &= \partial_t u_{h,j}(t)+\partial_t u_{p,j}(t)\\ &= [-\partial_x^n]u_{h,j}(t)+[-\partial_x^n]u_{p,j}(t)+F_j(t)\\ &= [-\partial_x^n]u_j(t)+F_j(t)\,.\end{aligned}$$

for $j=1,2$. So,

$$\partial_t u_1(t)-\partial_t u_2(t)=[-\partial_x^n]\{u_1(t)-u_2(t)\}+\{F_1(t)-F_2(t)\}\,.$$

Taking norm, we obtain

$$\begin{aligned}&\|\partial_t u_1(t)-\partial_t u_2(t)\|_{s-n}\\ &\quad=\|[-\partial_x^n]\{u_1(t)-u_2(t)\}+\{F_1(t)-F_2(t)\}\|_{s-n}\\ &\quad\leq\|[-\partial_x^n]\{u_1(t)-u_2(t)\}\|_{s-n}+\|F_1(t)-F_2(t)\|_{s-n}.\end{aligned}$$

Using $\|\partial_x^n f\|_{s-n}\leq\|f\|_s$, $\forall\, f\in H_{per}^s$ and

$H_{per}^s \subset H_{per}^{s-1}\subset H_{per}^{s-2}\subset\cdots\subset H_{per}^{s-(n-1)}\subset H_{per}^{s-n}$,

we get

$$\begin{aligned}\|\partial_t u_1(t)-\partial_t u_2(t)\|_{s-n} &\leq \|u_1(t)-u_2(t)\|_s+\|F_1(t)-F_2(t)\|_{s-n}\\ &\leq \|u_1(t)-u_2(t)\|_s+sup_{t\in[0,T]}\|F_1(t)-F_2(t)\|_{s-n}\\ &\leq \|u_1-u_2\|_{\infty,s}+\|F_1-F_2\|_{\infty,s}\\ &\leq \|\psi_1-\psi_2\|_s+(T+1)\|F_1-F_2\|_{\infty,s}.\end{aligned}$$

■

Remark 3.1 Inequality (3.8) says that the solution of the non-homogeneous problem (P_n^F) continuously depends on the initial data and the non homogeneity F, in compact intervals.

Corollary 3.1 The problem (P_n^F) has a unique solution.

Proof: This follows by applying inequality (3.8) with $\psi_1 = \psi_2 = \varphi$ and $F_1 = F_2 = F$.
■

Corollary 3.2 The unique solution of (P_n^F) is

$$u(t) = \sum_{k=-\infty}^{+\infty} e^{-k^n t}\hat{\varphi}(k)\varphi_k + \int_0^t \sum_{k=-\infty}^{+\infty} e^{-k^n(t-\tau)}\hat{F}(k,\tau)\varphi_k \, d\tau,$$

where $\varphi_k(x) := e^{ikx}$ for $x \in \mathrm{R}$.

Remark 3.2 Thus, Theorems 3.1, 3.2 and Corollaries 3.1, 3.2 are valid for the case n=4:

$$u_t + \partial_x^4 u = F(t);$$

whose homogeneous case (that is, F=0) was studied in [6].

Remark 3.3 Results close to Theorems 3.1, 3.2 and Corollaries 3.1, 3.2 are obtained for the odd order equation (P_n^F), where the homogeneous problem (F=0) is conservative; see [7]. And the solution non-homogeneous is

$$\boldsymbol{V_*^F(t) = V(t)\varphi + V_p(t)}$$

with

$$\boldsymbol{V(t)\varphi = \left[\left(e^{-(ik)^n t}\hat{\varphi}(k)\right)_{k\in Z}\right]^{\vee}, t \in R}$$

for the initial data $\boldsymbol{\varphi \in H_{per}^s}$, and

$$\boldsymbol{V_p(t) := \int_0^t V(t-\tau)F(\tau)d\tau}\,.$$

Note that $\boldsymbol{(ik)^n = \pm ik^n}$; that is, if $\boldsymbol{n-1}$ is multiple of four then $\boldsymbol{(ik)^n = ik^n}$ and otherwise $\boldsymbol{(ik)^n = -ik^n}$.

CONCLUSIONS

From our study of the n-th order equation (P_n^F), with n an even number multiple of four, in periodic Sobolev spaces, we have obtained the following results:

In Theorem 3.1 we obtain a particular solution of (P_n^F) using calculus in H_{per}^s and Sobolev inmersion.

In Theorem 3.2, using the Fourier transform, we proved that the non-homogeneous problem (P_n^F) is locally well posed in compacts, obtaining continuous dependence with respect to the initial data and the non homogeneity.

Finally, we gave some remarks about the n-th order equation when n is odd number.

References

[1] Iorio, R. and Iorio, V. Fourier Analysis and partial Differential Equations. Cambridge University, 2002.

[2] Liu, Z. and Zheng, S. Semigroups associated with dissipative system. Chapman Hall/CRC, 1999.

[3] Pathack, H. K. An introduction to Nonlinear analysis and fixed point theory. Springer, Singapur, 2018.

[4] Santiago Ayala, Y. and Rojas, S. Regularity and wellposedness of a problem to one parameter and its behavior at the limit, Bulletin of the Allahabad Mathematical Society, 32(02)(2017), 207-230.

[5] Santiago Ayala, Y. and Rojas, S. Existencia y regularidad de solución de la ecuación del calor en espacios de Sobolev periódico, Selecciones Matemáticas, 06(01)(2019), 49-65.

[6] Santiago Ayala, Y. Wellposedness of a Cauchy problem associated to even order equation, European Journal of Applied Sciences, 12(06)(2024), 512-530.

[7] Santiago Ayala, Y. Existence and continuous dependence of the local solution of non homogeneous third order equation and generalizations, Transactions on Machine Learning and Artificial Intelligence, 10(5)(2022), 43-56.