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Wellposedness of a Cauchy Problem Associated to the Even Order Equation

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ABSTRACT

In this article we prove that the Cauchy problem associated to n-th order equation in periodic Sobolev spaces is globally well posed when n is an even number multiple of four. We do this in an intuitive way using Fourier theory and in a fine version using semigroups theory. Finally, we demonstrate the dissipative property of the Cauchy problem using differential calculus in H^s_{per} .

Keywords: Semigroups theory, even order equation, existence of solution, dissipative property, Periodic Sobolev spaces, Fourier Theory.

MSC 2010: 35G10, 35Q53, 47D06, 35B40

INTRODUCTION

We study the problem:

$$(P_1): \partial_t u + \partial_x^n u = 0 \text{ in } H^{s-n}_{per} \text{, with } u(0) = \varphi \in H^s_{per} \text{,}$$

considering s a real number, n is an even number multiple of four and denoting by H^s_{per} to the periodic Sobolev space. This problem was proposed in [9], specifically in remark 4.3. In [9], the case n=3 was studied and some comments on the possibility of its generalization were given.

Also we can cite [6], where we find some works related to the model (P_1) coupled to Kuramoto-Sivashinski equation, and [1] where our model is stated among the proposed problems. In these additional references we have motivation to study the problem and inspiration with the ideas we find there.

We also cite some works about existence by semigroups [2], [3], [4] and take support in some results of [5].

Our article is organized as follows. In section 2, we indicate the methodology used and cite the references used. In section 3, we prove that problem (P_1) is well posed. Moreover, we introduce a family of operators that form a semigroup of class C_0 to state the result Theorem 3.3 and prove it in a fine version. In section 4, we study the dissipative property of the homogeneous problem (P_1) and applications. Finally, in section 5, we give the conclusions of our study.

METHODOLOGY

As a theorical framework in this article we use [6]. Also, we use the references [1], [8], [6] and [7] for the Fourier theory in periodic Sobolev spaces, and differential and integral calculus in Banach spaces.

THE PROBLEM (P_1) IS WELL POSED

We prove that (P_1) is well posed. Also, we introduce a family of operators that form a semigroup of class C_o, as we make it in Theorem 3.2. Finally, we state the Theorem 3.3 whose content is a fine version of Theorem 3.1 based on the semigroup $\{S(t)\}_{t\geq 0}$.

Theorem 3.1

Let s a fixed real number, n an even number multiple of four and the problem

$$(P_1)\left|\begin{array}{c} u \in C([0,\infty), H^s_{per}) \\ \partial_t u + \partial_x^n u = 0 \ \in \ H^{s-n}_{per} \\ u(0) = \varphi \ \in \ H^s_{per} \end{array}\right.$$

then (P_1) *is globally well posed, that is* $\exists!\, u \in C([0,\infty), H^s_{per}) \cap C^1((0,\infty), H^{s-n}_{per})$ *satisfying equation* (P_1) *so that the application* : $\varphi \rightarrow u$ *, which to every initial data* φ *assigns the solution u of the IVP* (P_1)*, is continuous. That is, for* φ *and* $\tilde{\varphi}$ *initial data close in* H^s_{per} *, their corresponding solutions u and* $\tilde{u}$ *respectively, are also close in the solution space.*

Also,

$$\|u(t) - \tilde{u}(t)\|_s \leq \|\varphi - \tilde{\varphi}\|_s, \ \forall t \in [0,\infty)$$

and

$$sup_{t\in[0,\infty)} \|u(t) - \tilde{u}(t)\|_s = \|\varphi - \tilde{\varphi}\|_s .$$

Moreover, the solution u satisfies

$$u(t) \in H^r_{per}, \ \forall t \in [0,\infty), \ \forall r \leq s$$

with

$$\|u(t)\|_s \leq \|\varphi\|_s \text{ and } \|u(t)\|_r \leq \|\varphi\|_s, \ \forall r < s \text{ and } t \in [0,\infty).$$

The application: $\varphi \rightarrow \partial_t u$*, wich for every initial data* φ *assigns the derivative of solution u of the IVP* (P_1) *is continuous. That is, for* φ *and* $\tilde{\varphi}$ *initial data close in* H^s_{per} *their corresponding* $\partial_t u$ *and* $\partial_t \tilde{u}$ *, respectively, are also close in the solution space. Also, the following inequalities are verified*

$$\|\partial_t u(t) - \partial_t \tilde{u}\ (t)\|_{s-n} \leq \|\varphi - \tilde{\varphi}\|_s, \ \forall t \in (0,\infty),$$

$$sup_{t\in(0,\infty),} \|\partial_t u(t) - \partial_t \tilde{u}\ (t)\|_{s-n} \leq \|\varphi - \tilde{\varphi}\|.$$

Moreover, $\|\partial_t u(t)\|_{s-n} \leq \|\varphi\|_s$, $\forall t \in (0, \infty)$.

Proof:
We prove it in the following way.

1. First, we obtain the candidate to the solution. In order to get it we apply the Fourier transformation to the equation

$$\partial_t u = -\partial_x^n u$$

and using $(ik)^n = k^n$, $\forall\, k \in Z$, we have

$$\begin{aligned}\partial_t \hat{u} &= -(ik)^n \hat{u}\\ &= -k^n \hat{u}\end{aligned}$$

which for every k is an ODE with initial data $\hat{u}(k, 0) = \hat{\varphi}(k)$.

Thus, solving the IVP's

$$(\Omega_k)\quad \left|\begin{array}{c} \hat{u} \in C\big([0,\infty), l_s^2(Z)\big) \\ \partial_t \hat{u}(k,t) = -k^n\, \hat{u}(k,t) \\ \hat{u}(k,0) = \hat{\varphi}(k) \end{array}\right.$$

we obtain

$$\hat{u}(k,t) = e^{-k^n t}\hat{\varphi}(k),$$

from which we get our candidate to the solution:

$$\begin{aligned} u(t) &= \sum_{k=-\infty}^{+\infty} \hat{u}(k,t)\varphi_k \\ &= \sum_{k=-\infty}^{+\infty} e^{-k^n t}\hat{\varphi}(k)\varphi_k \quad (3.1)\end{aligned}$$

here we are denoting $\varphi_k(x) = e^{ikx}$ for $x \in R$.

2. Second, we prove:

$$u(t) \in H_{per}^s \text{ and } \|u(t)\|_s \leq \|\varphi\|_s \tag{3.2}$$

In effect, let $t \in (0,\infty)$, $\varphi \in H_{per}^s$, we have

$$\|u(t)\|^2_{H^s_{per}} = 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s e^{-2k^n t} |\hat{\varphi}(k)|^2$$
$$\leq 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\hat{\varphi}(k)|^2 < \infty \quad (3.3)$$
$$= \|\varphi\|^2_{H^s_{per}} .$$

Obviously, it holds (3.2) for $t = 0$.

3. We will prove that $u(\cdot)$ is continuous in $[0,\infty)$. Let $t' \in [0,\infty)$,

$$\|u(t) - u(t')\|^2_{H^s_{per}}$$
$$= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \left|\left(e^{-k^n t} - e^{-k^n t'}\right)\hat{\varphi}(k)\right|^2$$
$$= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\hat{\varphi}(k)|^2 \, |H(t)|^2 \quad (3.4)$$

where $H(t) := e^{-k^n t} - e^{-k^n t'}$. We see that $\lim_{t\to t'} H(t) = 0$.

In order to interchange limits, we need the uniform convergence of the series. For this, we take the k-th term of the series and bound it by a convergent series, that is

$$I_{k,t} := 2\pi(1+k^2)^s |\hat{\varphi}(k)|^2 \left|e^{-k^n t} - e^{-k^n t'}\right|^2$$
$$\leq 8\pi(1+k^2)^s |\hat{\varphi}(k)|^2,$$

there we have used the triangular inequality (property of the norm) and the inequality $e^{-\theta} \leq 1$ for $\theta \in [0,\infty)$. Thus,

$$\sum_{k=-\infty}^{+\infty} I_{k,t} \leq 4\|\varphi\|^2_{H^s_{per}} < \infty ,$$

and using the M-Test of Weierstrass Theorem, we have the series converges uniformly. Now, we can interchange limits, that is

$$\lim_{t\to t'} \|u(t) - u(t')\|^2_{H^s_{per}} = \sum_{k=-\infty}^{+\infty} \underbrace{\lim_{t\to t'} I_{k,t}}_{=0} = 0$$

and then we conclude

$$\lim_{t\to t'}\|u(t)-u(t')\|_{H^s_{per}} = 0\,.$$

4. We will prove

$$\left\|\frac{u(t+h)-u(t)}{h}+\partial_x^n u(t)\right\|_{H^{s-n}_{per}} \to 0 \quad \text{when } h\to 0.$$

In effect, let t∈R,

$$\left\|\frac{u(t+h)-u(t)}{h}+\partial_x^n u(t)\right\|^2_{H^{s-n}_{per}}$$
$$= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s-n}|\hat{\varphi}(k)|^2\left|\frac{e^{-k^n(t+h)}-e^{-k^n t}}{h}+(k)^n e^{-k^n t}\right|^2$$
$$= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s-n}|\hat{\varphi}(k)|^2\left|e^{-k^n t}.M(h)\right|^2 \quad (3.5)$$

where $M(h) := \left\{\frac{e^{-k^n h}-1}{h}+k^n\right\}$.

Using L'Hospital we have $M(h)\to 0$ when $h\to 0$. Again, to interchange limits, we need the uniform convergence of the series. For this we will bound the k-th term of the series. Previously, for $h\neq 0$, we observe

$$\frac{e^{-k^n h}-1}{h} = \int_0^h \frac{1}{h}\frac{\partial}{\partial s}\left\{e^{-k^n s}\right\}ds$$
$$= \int_0^h \frac{1}{h}[-k^n]e^{-k^n s}ds$$

and taking norm, we have

$$\left|\frac{e^{-k^n h}-1}{h}\right| \le \frac{1}{h}|-k^n|\int_0^h\left|e^{-k^n s}\right|ds$$
$$\le \frac{1}{h}|k|^n.h \;=\; |k|^n,$$

for h>0. That is,

$$\left|\frac{e^{-k^n h}-1}{h}\right| \le |k|^n \quad (3.6)$$

for h>0. If h<0, (3.6) also holds. Using inequality (3.6), we are going to bound $|M\ (h)|^2$ as follows:

$$\begin{aligned} |M(h)|^2 &\leq \{2|k|^n\}^2 \\ &\leq 4\{|k|^2\}^n \\ &\leq 4\{1+|k|^2\}^n \end{aligned} \tag{3.7}$$

Let us bound the k-th term of the series. Here we will use the estimation (3.7)

$$\begin{aligned} (1+k^2)^{s-n}|\hat{\varphi}(k)|^2 e^{-2k^n t}|M\ (h)|^2 &\leq 4(1+k^2)^{s-n}|\hat{\varphi}(k)|^2\{1+|k|^2\}^n \\ &= 4(1+k^2)^s|\hat{\varphi}(k)|^2 \end{aligned}$$

and, since $2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^s|\hat{\varphi}(k)|^2 = \|\varphi\|_s^2 < \infty$ for $\varphi \in H_{per}^s$ using the M-Test of Weierstrass we get the series (3.5) converges uniformly and then it is possible to interchange limits and obtain

$$\left\|\frac{u(t+h)-u(t)}{h} + \partial_x^n u(t)\right\|_{s-n}^2 \rightarrow 0 \text{ when } h \rightarrow 0\,. \tag{3.8}$$

5. We will prove the continuous dependency of the solution with respect to the initial data, that is, let φ and $\tilde{\varphi}$ be close data in H_{per}^s , then their corresponding solutions u *and* $\tilde{u}$, respectively, are also close in the solution space. Let $t \in [0,\infty)$,

$$\begin{aligned} \|u(t)-\tilde{u}(t)\|_{H_{per}^s}^2 &= 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^s\left|e^{-k^n t}\left(\hat{\varphi}(k)-\hat{\tilde{\varphi}}(k)\right)\right|^2 \\ &\leq 2\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^s\ \left|\hat{\varphi}(k)-\hat{\tilde{\varphi}}(k)\right|^2 \\ &= \|\varphi-\tilde{\varphi}\|_{H_{per}^s}^2 \end{aligned} \tag{3.9}$$

Taking supremum over $[0,\infty)$ we have

$$sup_{t\in[0,\infty)}\|u(t)-\tilde{u}(t)\|_{H_{per}^s} = \|\varphi-\tilde{\varphi}\|_{H_{per}^s} \tag{3.10}$$

Hence, we have: if $\varphi \rightarrow \tilde{\varphi}$ then $u \rightarrow \tilde{u}$.

6. Uniqueness of Solution. Equality (3.10) or (3.9) will allow us to prove the solution is unique. In effect, let $\varphi \in H_{per}^s$ and suppose there are u and $\tilde{u}$ two solutions, then using (3.10) we have,

$$\|u(r)-\tilde{u}(r)\|_{H_{per}^s} = sup_{t\in[0,\infty)}\|u(t)-\tilde{u}(t)\|_{H_{per}^s} = \|\varphi-\varphi\|_{H_{per}^s} = 0, \forall r \in [0,\infty)$$

from where we conclude that $u = \tilde{u}$. Thus, problem (P_1) is well posed and its unique solution, which depends continuously on the initial data, is

$$u(t) = \sum_{k=-\infty}^{+\infty} e^{-k^n t}\hat{\varphi}(k)\varphi_k.$$

7. Now, we consider the case $r < s$. Here we have $H_{per}^s \subset H_{per}^r$ and since the initial data $\varphi \in H_{per}^s$, then $\varphi \in H_{per}^r$ and satisfies

$$\|\varphi\|_r \leq \|\varphi\|_s. \tag{3.11}$$

From (3.3) for r and using (3.11) we get

$$\|u(t)\|_r^2 = \|\varphi\|_r^2 \leq \|\varphi\|_s^2 < \infty .$$

That is,

$$u(t) \in H_{per}^r ,\ \forall r \in (-\infty, s). \tag{3.12}$$

The case $r = s$ was already proven on the item 2.

Therefore, from (3.2) and (3.12), we conclude for $t \in [0,\infty)$

$$u(t) \in H_{per}^r ,\ \forall r \in (-\infty, s].$$

8. We will prove that $\partial_t u\,(\cdot)$ is continuous in $(0,\infty)$. Let $t, t' \in (0,\infty)$, using the inequality $\|\partial_x^m u(t)\|_{s-m} \leq \|u(t)\|_s$ and continuity of $u\,(\cdot)$, we obtain

$$\begin{aligned}\|\partial_t u(t) - \partial_t u(t')\|_{s-n} &= \|-\partial_x^n u(t)+\partial_x^n u(t')\|_{s-n}\\ &= \|\partial_x^n (u(t) - u(t'))\|_{s-n}\\ &\leq \|u(t) - u(t')\|_s \ \to 0\end{aligned} \tag{3.13}$$

when $t \to t'$. That is, $\partial_t u \in C((0,\infty) , H_{per}^{s-n})$.

9. Let $\varphi \in H_{per}^r$, if define $W(t)\varphi := \sum_{k=-\infty}^{+\infty}(-k^n)e^{-k^n t}\hat{\varphi}(k)\varphi_k$ then $W(t)\varphi \in H_{per}^{s-n}$ and $\|W(t)\varphi\|_{s-n} \leq \|\varphi\|_s$, $\forall t \in (0,\infty)$. That is, $W(t) \in L(H_{per}^s, H_{per}^{s-n})$ with $\|W(t)\| \leq 1$. In effect, using $|-k^n|^2 = |k^n|^2 \leq (|k|^n)^2 = (|k|^2)^n \leq (1 + |k|^2)^n$, $\forall k \in Z$ and $e^{-\theta} \leq 1$, $\forall \theta \in [0,\infty)$, we have

$$\begin{aligned}\|W(t)\varphi\|_{s-n}^2 &= 2\pi \sum_{k=-\infty}^{+\infty}(1 + k^2)^{s-n}\left|(-k^n)e^{-k^n t}\hat{\varphi}(k)\right|^2\\ &= 2\pi \sum_{k=-\infty}^{+\infty}(1 + k^2)^{s-n}\,|(k^n)|^2|\hat{\varphi}(k)|^2\\ &\leq 2\pi \sum_{k=-\infty}^{+\infty}(1 + k^2)^{s}\,|\hat{\varphi}(k)|^2 \ < \infty\end{aligned}$$

$$= \|\varphi\|_s^2 .$$

10. From item 4 and 9, we have $\partial_t u(t) = W(t)\varphi$.

■

Next, we have the following result

Corollary 3.1:
The unique solution of (P_1) is

$$u(t) = \sum_{k=-\infty}^{+\infty} e^{-k^n t}\hat{\varphi}(k)\varphi_k ,$$

where $\varphi_k(x) := e^{ikx}$ *for* $x \in R$.

Corollary 3.2:
With the hypothesis of preceding Theorem, we obtain

1. $u \in C([0,\infty), H_{per}^r) \cap C^1((0,\infty), H_{per}^{r-n})$, $\forall r < s$.
2. *u satisfies*

$$\|u(t)\|_r \leq \|\varphi\|_s , \forall t \in [0,\infty), \forall r < s , \tag{3.14}$$

$$\|\partial_t u(t)\|_{r-n} \leq \|\varphi\|_s , \forall t \in (0,\infty), \forall r < s . \tag{3.15}$$

3. *That is,*

$$\|u(t) - \tilde{u}(t)\|_r \leq \|\varphi - \tilde{\varphi}\|_s , \forall t \in [0,\infty), \forall r < s ,$$

$$sup_{t\in[0,\infty)} \|u(t) - \tilde{u}(t)\|_r \leq \|\varphi - \tilde{\varphi}\|_s , \forall r < s .$$

4. *Moreover,*

$$\|\partial_t u(t) - \partial_t \tilde{u}(t)\|_{r-n} \leq \|\varphi - \tilde{\varphi}\|_s , \forall t \in (0,\infty), \forall r \leq s ,$$

$$sup_{t\in(0,\infty)} \|\partial_t u(t) - \partial_t \tilde{u}(t)\|_{r-n} \leq \|\varphi - \tilde{\varphi}\|_s , \forall r < s .$$

Proof:
The inequality (3.14) follows of the Sobolev continuous imbedding.

We will use the Sobolev continuous Imbedding and item 9 for prove that if $\varphi \in H_{per}^s$ then $W(t)\varphi \in H_{per}^{r-n}$ and $\|W(t)\varphi\|_{r-n} \leq \|\varphi\|_s$, $\forall t \in (0,\infty)$, $\forall r < s$. That is, $W(t) \in L(H_{per}^s, H_{per}^{r-n})$ with $\|W(t)\| \leq 1$, $\forall r < s$.

In effect, using $|k^n|^2 \leq (1+|k|^2)^n$, $\forall k \in Z$ and $e^{-\theta} \leq 1$, $\forall \theta \in [0,\infty)$, we have

$$\begin{aligned}\|W(t)\varphi\|_{r-n}^2 &= 2\pi \sum_{k=-\infty}^{+\infty}(1+k^2)^{r-n}\left|(-k^n)e^{-k^n t}\hat{\varphi}(k)\right|^2 \\ &\leq 2\pi \sum_{k=-\infty}^{+\infty}(1+k^2)^{r-n}|(k^n)|^2|\hat{\varphi}(k)|^2 \\ &\leq 2\pi \sum_{k=-\infty}^{+\infty}(1+k^2)^{r}|\hat{\varphi}(k)|^2 \\ &\leq 2\pi \sum_{k=-\infty}^{+\infty}(1+k^2)^{s}|\hat{\varphi}(k)|^2 < \infty \\ &= \|\varphi\|_s^2\end{aligned} \tag{3.16}$$

■

Now, we will introduce a family of operators which verify the condition of being a semigroup of class C_0.

Theorem 3.2

Let be $s \in R$ and n an even number multiple of four. The application

$$S_n : [0,\infty) \to L\,(H_{per}^s)$$

$$t \to S(t)$$

such that $S_n(t) = e^{-(\partial_x^n)t}$, that is, applies

$$S_n(t)\varphi = \left[\left(e^{-k^n t}\hat{\varphi}(k)\right)_{k\in Z}\right]^{\vee}, \forall\, \varphi \in H_{per}^s,$$

then $\{S_n(t)\}_{t\geq 0}$ is a contraction semigroup of class C_0 on H_{per}^s. Thus $\{S_n\}_{n\in M}$ is a family of semigroups on H_{per}^s, where

$$M := \{n \in N \;/\; n \text{ is even number multiple of four}\}.$$

And for simplicity we will denote to S_n as S.

Moreover, the following assertions hold:

1. *If $\varphi \in H_{per}^s$ then $S(\cdot)\,\varphi \in C([0,\infty), H_{per}^s)$.*
2. *The application $\varphi \to S(\cdot)\,\varphi$ is continuous and verifies:*

$$\|S(t)\psi_1 - S(t)\psi_2\|_{H_{per}^s} \leq \|\psi_1 - \psi_2\|_{H_{per}^s}, \forall t \in [0,\infty)$$

and

$$sup_{t\in[0,\infty)}\|S(t)\psi_1 - S(t)\psi_2\|_{H_{per}^s} = \|\psi_1 - \psi_2\|_{H_{per}^s}$$

with $\psi_i \in H_{per}^s$ for i = 1, 2.

3. *If* $\varphi \in H^s_{per}$ *then* $\partial_t S(t)\varphi \in H^{s-n}_{per}$ *and* $\|\partial_t S(t)\varphi\|_{s-n} \leq \|\varphi\|_s$, $\forall t \in (0,\infty)$. *That is,* $\partial_t S(t) \in L(H^s_{per}, H^{s-n}_{per})$, $\forall t \in (0,\infty)$, *where*

$$\partial_t S(t)\varphi = \left[\left((-k^n)e^{-k^n t}\hat{\varphi}(k)\right)_{k\in Z}\right]^{\vee} \in H^{s-n}_{per}, \forall\, \varphi \in H^s_{per}.$$

4. *If* $\varphi \in H^s_{per}$ *then* $\partial_t S(\cdot)\,\varphi \in C((0,\infty), H^{s-n}_{per})$.
5. *The application:* $\psi \to \partial_t S(\cdot)\,\psi$ *is continuous and verifies:*

$$\|\partial_t S(t)\psi_1 - \partial_t S(t)\psi_2\|_{H^{s-n}_{per}} \leq \|\psi_1 - \psi_2\|_{H^s_{per}}, \forall t \in (0,\infty)$$

and

$$sup_{t\in(0,\infty)} \|\partial_t S(t)\psi_1 - \partial_t S(t)\psi_2\|_{H^{s-n}_{per}} \leq \|\psi_1 - \psi_2\|_{H^s_{per}}$$

with $\psi_i \in H^s_{per}$ *for i* = 1, 2.

Proof:

We first observe $S(0)\varphi = [(\hat{\varphi}(k))_{k\in Z}]^{\vee} = [\hat{\varphi}]^{\vee} = \varphi, \forall\varphi \in H^s_{per}$, thus $S(0) = I$. From linearity of Fourier transformation and its inverse, we have that $S(t)$ is linear. In effect, let be $a \in \mathbb{C}$, φ, $\psi \in H^s_{per}$, we have

$$\begin{aligned} S(t)(a\varphi + \psi) &= \left[\left(e^{-k^n t}[a\varphi + \psi]^\wedge(k)\right)_{k\in Z}\right]^{\vee} \\ &= \left[\left(e^{-k^n t}\left[a\hat{\varphi}(k) + \hat{\psi}(k)\right]\right)_{k\in Z}\right]^{\vee} \\ &= \left[a\left(e^{-k^n t}\hat{\varphi}(k)\right)_{k\in Z} + \left(e^{-k^n t}\hat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\ &= a\left[\left(e^{-k^n t}\hat{\varphi}(k)\right)_{k\in Z}\right]^{\vee} + \left[\left(e^{-k^n t}\hat{\psi}(k)\right)_{k\in Z}\right]^{\vee} \\ &= aS(t)(\varphi) + S(t)(\psi)\,, \end{aligned}$$

for $t \in [0,\infty)$.

If $\varphi \in H^s_{per}$ and $t \in (0,\infty)$, we will prove that $S(t)\varphi \in H^s_{per}$ and $\|S(t)\varphi\|_s = \|\varphi\|_s$, that is $\|S(t)\| = 1$.

In effect, similar to (3.3), for $t \in (0,\infty)$ and $\varphi \in H^s_{per}$ we have

$$\begin{aligned} \|S(t)\varphi\|^2_{H^s_{per}} &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s \left|e^{-k^n t}\hat{\varphi}(k)\right|^2 \\ &\leq 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^s |\hat{\varphi}(k)|^2 \\ &= \|\varphi\|^2_{H^s_{per}} < \infty\,. \end{aligned}$$

Then, $S(t)\varphi \in H^s_{per}$ and $\|S(t)\varphi\|_s \leq \|\varphi\|_s$, $\forall t \in (0,\infty)$, that is $S(t) \in L\,(H^s_{per})$ with $\|S(t)\| \leq 1$, $\forall t \in (0,\infty)$.

Therefore,

$$\|S(t)\varphi\|_s \leq \|\varphi\|_s,\ \forall t \in [0,\infty),\ \forall \varphi \in H^s_{per}\,. \tag{3.17}$$

That is,

$$S(t) \in L\,(H^s_{per}\,) \text{ with } \|S(t)\| \leq 1,\ \forall t \in [0,\infty). \tag{3.18}$$

Now we will prove that $S\,(t + r) = S(t) \circ S(r)$, $\forall t, r \in [0,\infty)$. In effect, let be $f \in H^s_{per}$ and $t, r \in (0,\infty)$,

$$\begin{aligned} S(t+r)f &= \left[\left(e^{-k^n(t+r)}\hat{f}(k)\right)_{k\in Z}\right]^\vee \\ &= \left[\left(e^{-k^n t}e^{-k^n r}\hat{f}(k)\right)_{k\in Z}\right]^\vee \end{aligned} \tag{3.19}$$

We know that if $f \in H^s_{per}$ then $\hat{f} \in l^2_s$, that is

$$\textstyle\sum_{k=-\infty}^{+\infty}(1+k^2)^s\left|\hat{f}(k)\right|^2 < \infty\,. \tag{3.20}$$

We affirm that

$$\left(e^{-k^n r}\hat{f}(k)\right)_{k\in Z} \in l^2_s\,, \forall r \in [0,\infty)\,. \tag{3.21}$$

Indeed, when $r = 0$ it is evident that the statement is true. Thus, we will prove the case $r \in (0,\infty)$. For this, it is enough to observe that

$$\begin{aligned} &\sum_{k=-\infty}^{+\infty}(1+k^2)^s\left|e^{-k^n r}\hat{f}(k)\right|^2 \\ &= \sum_{k=-\infty}^{+\infty}(1+k^2)^s\,\underbrace{\left|e^{-2k^n r}\right|}_{\leq 1}\left|\hat{f}(k)\right|^2 \\ &\leq \sum_{k=-\infty}^{+\infty}(1+k^2)^s\left|\hat{f}(k)\right|^2 < \infty\,, \end{aligned}$$

since it worth (3.20).

Then, from (3.21) and taking the inverse Fourier transform, we have

$$\left[\left(e^{-k^n r}\hat{f}(k)\right)_{k\in Z}\right]^\vee \in H^s_{per}\,, \forall r \in [0,\infty)\,.$$

This motivates us to define

$$\left[g_r = \left(e^{-k^n r}\hat{f}(k)\right)_{k\in Z}\right]^{\vee} \in H^s_{per} .$$

That is,

$$g_r = S(r)f. \tag{3.22}$$

Also, taking the Fourier transform to g_r we obtain

$$\widehat{g_r} = \left(e^{-k^n r}\hat{f}(k)\right)_{k\in Z} ,$$

that is

$$\widehat{g_r}(k) = e^{-k^n r}\hat{f}(k) \, , \forall k \in Z \, . \tag{3.23}$$

Using (3.23) in (3.19) and from (3.22) we have

$$\begin{aligned} S(t+r) &= \left[\left(e^{-k^n t}\widehat{g_r}(k)\right)_{k\in Z}\right]^{\vee} \\ &= S(t)g_r \\ &= S(t)[S(r)f] \\ &= [S(t)\circ S(r)]f \, , \forall t, r \in (0,\infty). \end{aligned}$$

Thus,

$$s(t+r) = S(t)\circ S(r), \forall t, r \in (0,\infty) \, . \tag{3.24}$$

If t = 0 or r = 0, then the equality of (3.24) is also true, with this we conclude the proof of

$$S\,(t+r) = S(t)\circ S(r), \forall t, r \in [0,\infty). \tag{3.25}$$

Now, we will prove the continuity of $t \to S(t)\varphi$

$$\|S(t+h)\varphi \, - \, S(t)\varphi\|_{H^s_{per}} \to 0 \text{ when } h \to 0. \tag{3.26}$$

In effect, using item 3 of the proof of the preceding theorem, we have

$$\begin{aligned} &\|S(t+h)\varphi \, - \, S(t)\varphi\|^2_{H^s_{per}} \\ &= 2\pi \sum_{k=-\infty}^{+\infty}(1+k^2)^s\left|\left(e^{-k^n(t+h)} - e^{-k^n t}\right)\hat{\varphi}(k)\right|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty}(1+k^2)^s|\hat{\varphi}(k)|^2 \, |H(t+h)|^2 \end{aligned} \tag{3.27}$$

where $H(t+h) := e^{-k^n(t+h)} - e^{-k^n t}$.

We observe that $\lim_{h\to 0} H(t+h) = 0$.

Now, we again need the uniform convergence of the series in order to interchange the limits. For this, we take the k-th term of the series and bound it with a convergent series, that is

$$I_{k,t,h} := 2\pi(1+k^2)^s|\hat{\varphi}(k)|^2\left|e^{-k^n(t+h)} - e^{-k^n t}\right|^2$$
$$\leq 8\pi(1+k^2)^s|\hat{\varphi}(k)|^2,$$

where we have used the triangular inequality (property of the norm) and the inequality $e^{-\theta} \leq 1$ for $\theta \in [0,\infty)$.

Thus,

$$\sum_{k=-\infty}^{+\infty} I_{k,t,h} \leq 4\|\varphi\|_s^2 < \infty, \tag{3.28}$$

and using the M-Test Weierstrass Theorem we get the series in (3.28) converges uniformly. Then, interchanging limits is allowed, that is

$$\lim_{h\to 0}\|S(t+h)\varphi - S(t)\varphi\|_s^2 = \sum_{k=-\infty}^{+\infty} \underbrace{\lim_{h\to 0} I_{k,t,h}}_{=0} = 0$$

and from here we conclude

$$\lim_{h\to 0}\|S(t+h)\varphi - S(t)\varphi\|_s = 0.$$

Remark 3.1: *It verifies*

$$\lim_{h\to 0}\|S(t)\varphi - \varphi\|_s = 0, \forall \varphi \in H_{per}^s.$$

Remark 3.2: *With the remark 3.1 we would have that $\{S(t)\}_{t\geq 0}$ is a semigroup of class C_0.*

Let ψ_1 and ψ_2 close data in H_{per}^s , then we will prove that their corresponding $S(\cdot)\psi_1$ and $S(\cdot)\psi_2$, respectively, are also close. Since $\{S(t)\}_{t\geq 0}$ is a semigroup of class C_0 of contraction, for $t \in [0,\infty)$, we have

$$\|S(t)\psi_1 - S(t)\psi_2\|_{H_{per}^s} = \|S(t)(\psi_1 - \psi_2)\|_{H_{per}^s} \leq \|\psi_1 - \psi_2\|_{H_{per}^s}.$$

Taking supremum over $[0,\infty)$ we have

$$sup_{t\in[0,\infty)}\|S(t)\psi_1 - S(t)\psi_2\|_{H_{per}^s} = \|\psi_1 - \psi_2\|_{H_{per}^s}. \tag{3.29}$$

From here we have that if $\psi_1 \to \psi_2$ then $S(\cdot)\psi_1 \to S(\cdot)\psi_2$.

We will prove: If $\varphi \in H^s_{per}$ then $\partial_t S(t)\varphi \in H^{s-n}_{per}$ and $\|\partial_t S(t)\varphi\|_{s-n} \leq \|\varphi\|_s$. In effect, using $|k^n|^2 \leq (|k|^n)^2 = (|k|^2)^n \leq (1 + |k|^2)^n$, $\forall k \in Z$ and $e^{-\theta} \leq 1$, $\forall \theta \in [0,\infty)$, we have

$$\begin{aligned}\|\partial_t S(t)\varphi\|^2_{s-n} &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} \left|(-k^n)e^{-k^n t}\hat{\varphi}(k)\right|^2 \\ &\leq 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} |k^n|^2 |\hat{\varphi}(k)|^2 \\ &\leq 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s} |\hat{\varphi}(k)|^2 < \infty \\ &= \|\varphi\|^2_s .\end{aligned}$$

That is, $\|\partial_t S(t)\varphi\|_{s-n} \leq \|\varphi\|_s$. From this inequality we obtain

$$\|\partial_t S(t)\phi_1 - \partial_t S(t)\phi_2\|_{s-n} \leq \|\phi_1 - \phi_2\|_s,$$

with $\phi_i \in H^s_{per}$ for $i = 1, 2$.

So, taking supremum over $(0,\infty)$, we have

$$sup_{t\in(0,\infty),} \|\partial_t S(t)\phi_1 - \partial_t S(t)\phi_2\|_{s-n} \leq \|\phi_1 - \phi_2\|_s.$$

Finally, if $\varphi \in H^s_{per}$ we will prove the continuity of $t \to \partial_t S(t)\varphi$. That is

$$\|\partial_t S\,(t+h)\,\varphi - \partial_t S(t)\varphi\|_{s-n} \to 0 \text{ when } h \to 0.$$

In effect, as item 3 of the proof of the preceding theorem, we have

$$\begin{aligned}&\|\partial_t S(t+h)\varphi - \partial_t S(t)\varphi\|^2_{s-n} \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} \left|\left(e^{-k^n(t+h)} - e^{-k^n t}\right).(-k^n)\hat{\varphi}(k)\right|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} \left|\left(e^{-k^n h} - 1\right)\left(e^{-k^n t}\right).(-k^n)\hat{\varphi}(k)\right|^2 \\ &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} \left|e^{-k^n h} - 1\right|^2 \left|e^{-k^n t}\right|^2 . |k^n|^2 |\hat{\varphi}(k)|^2 \\ &= 2\pi \textstyle\sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} |H(h)|^2 . e^{-2k^n t} |k^n|^2 |\hat{\varphi}(k)|^2 \end{aligned} \quad (3.30)$$

where $H(h) := e^{-k^n h} - 1$. We observe that $\lim_{h\to 0} H(h) = 0$.

Now, we again need the uniform convergence of the series in order to interchange the limits. For this, we take the k-th term of the series and bound it with a convergent series, that is

$$I_{k,t,h} = 2\pi(1+k^2)^{s-n}|H(h)|^2 . e^{-2k^n t} . |k^n|^2 |\hat{\varphi}(k)|^2$$
$$\leq 8\pi(1+k^2)^s |\hat{\varphi}(k)|^2 ,$$

where we have used the triangular inequality (property of the norm), $|k^n|^2 \leq (1+|k|^2)^n$, $\forall k \in Z$ and the equality $e^{-\theta} \leq 1$ for $\theta \in [0,\infty)$. Thus,

$$\sum_{k=-\infty}^{+\infty} I_{k,t,h} \leq 4\|\varphi\|_s^2 < \infty , \quad (3.31)$$

and using the M-Test Weierstrass Theorem we get the series in (3.31) converges uniformly. Then, interchanging limits is allowed, that is

$$\lim_{h\to 0}\|\partial_t S(t+h)\varphi - \partial_t S(t)\varphi\|_{s-n}^2 = \sum_{k=-\infty}^{+\infty} \underbrace{\lim_{h\to 0} I_{k,t,h}}_{=0} = 0$$

hence, we conclude

$$\lim_{h\to 0}\|\partial_t S(t+h)\varphi - \partial_t S(t)\varphi\|_{s-n} = 0 .$$

■

We will give some additional properties of $\{S(t)\}_{t\geq 0}$.

Corollary 3.3:

With the hypothesis of preceding Theorem, the following assertions hold

1. *If $\phi \in H_{per}^s$ then $S(t)\phi \in H_{per}^s$ and $\|S(t)\phi\|_r \leq \|\phi\|_s$, $\forall t \in [0,\infty)$, $\forall r < s$. That is $S(t) \in L(H_{per}^s, H_{per}^r)$, $\forall t \in [0,\infty)$, $\forall r < s$.*
2. *If $\phi \in H_{per}^s$ then $S(\cdot)\phi \in C([0,\infty), H_{per}^r)$, $\forall r < s$.*
3. *The application: $\phi \to S(\cdot)\phi$ is continuous and verifies:*

$$\|S(t)\phi_1 - S(t)\phi_2\|_r \leq \|\phi_1 - \phi_2\|_s, \forall t \in [0,\infty), \forall r < s,$$
$$sup_{t\in[0,\infty)} \|S(t)\phi_1 - S(t)\phi_2\|_r \leq \|\phi_1 - \phi_2\|_s, \forall r < s$$

   with $\phi_i \in H_{per}^s$ for i = 1, 2.

4. *If $\phi \in H_{per}^s$ then $\partial_t S(t)\phi \in H_{per}^{r-n}$ and $\|\partial_t S(t)\phi\|_{r-n} \leq \|\phi\|_s$, $\forall t \in (0,\infty)$, $\forall r < s$. That is $\partial_t S(t) \in L(H_{per}^s, H_{per}^{r-n})$, $\forall t \in (0,\infty)$, $\forall r < s$ where*

$$\partial_t S(t)\varphi = \left[\left((-k^n)e^{-ik^n t}\hat{\varphi}(k)\right)_{k\in Z}\right]^{\vee} \in H_{per}^{r-n}, \forall \varphi \in H_{per}^s , \forall r < s .$$

5. *If $\phi \in H_{per}^s$ then $\partial_t S(\cdot)\phi \in C((0,\infty), H_{per}^{r-n})$, $\forall r < s$.*

6. *The application:* $\phi \to \partial_t S(\cdot)\,\phi$ *is continuous and verifies:*

$$\|\partial_t S(t)\phi_1 - \partial_t S(t)\phi_2\|_{r-n} \leq \|\phi_1 - \phi_2\|_s,\ \forall t \in (0,\infty),\ \forall r < s,$$
$$sup_{t\in R}\ \|\partial_t S(t)\phi_1 - \partial_t S(t)\phi_2\|_{r-n} \leq \|\phi_1 - \phi_2\|_s,\ \forall r < s$$

with $\phi_i \in H^s_{per}$ *for* $i = 1, 2$.

Proof: Its proof is analogous to the proof of the second Corollary of Theorem 3.1, where we use Sobolev imbedding.

■

Following, we state the Theorem 3.1 in function of the semigroup $\{S(t)\}_{t\geq 0}$.

Theorem 3.3:
Let be $s \in R$, *n an even number multiple of four and* $\{S(t)\}_{t\geq 0}$ *the semigroup of class* C_0 *from Theorem 3.2, then* $S(\cdot)\varphi$ *is the unique solution of*

$$\left|\begin{array}{l} u \in C\big([0,\infty), H^s_{per}\big) \cap C^1\big((0,\infty), H^{s-n}_{per}\big) \\ \qquad u_t = Au \text{ in } H^{s-n}_{per} \\ \qquad u(0) = \varphi \in H^s_{per} \end{array}\right.$$

in the sense that

$$\lim_{h\to 0}\left\|\frac{S(t+h)\varphi - S(t)\varphi}{h} - AS(t)\varphi\right\|_{H^{s-n}_{per}} = 0 \tag{3.32}$$

where $A \coloneqq -\partial_x^n$, *and if* $\varphi_1 \sim \varphi_2$ *then* $S(\cdot)\varphi_1 \sim S(\cdot)\varphi_2$.

Moreover, the following regularity is satisfied: If $\varphi \in H^s_{per}$ *then* $S(t)\varphi \in H^r_{per}, \forall r \leq s, \forall t \in [0,\infty)$ *and* $\|S(t)\varphi\|_{H^r_{per}} \leq \|\varphi\|_{H^s_{per}}, \forall t \in [0,\infty), \forall r \leq s$. *Also,* $\|\partial_t S(t)\varphi\|_{H^{r-n}_{per}} \leq \|\varphi\|_{H^s_{per}}, \forall t \in (0,\infty), \forall r \leq s, \forall \varphi \in H^s_{per}$.

Proof:
The proof of (3.32) is analogous to the item 4 of the proof of Theorem 3.1. And the proof of the remaining statement is also similar to the proof of Theorem 3.1 and as a consequence of Theorem 3.2.

■

DISSIPATIVE PROPERTY OF THE PROBLEM (P_1)

Theorem 4.1
Let n is an even number multiple of four and w the solution of (P_1) *with initial data* $\varphi \in H^s_{per}$ *then we obtain the following results:*

1. $\partial_t\|w(t)\|^2_{s-n} = -4\pi\sum_{k=-\infty}^{+\infty}(1+k^2)^{s-n}k^n|\widehat{w}(k)|^2 \leq 0$.

2. $\|w(t)\|_{s-n} \leq \|\varphi\|_{s-n} \leq \|\varphi\|_{s}$, $t \in [0,\infty)$.

Proof:
As $H^s_{per} \subset H^{s-n}_{per}$ then the following expressions: $\langle \partial_t w, w\rangle_{s-n}$ and $\langle w, \partial_t w\rangle_{s-n}$ are well defined. So, we have

$$\begin{aligned} \partial_t \|w(t)\|^2_{s-n} &= \partial_t \langle w(t), w(t)\rangle_{s-n} \\ &= \langle \partial_t w(t), w(t)\rangle_{s-n} + \langle w(t), \partial_t w(t)\rangle_{s-n} \\ &= 2Re\, \langle \partial_t w(t), w(t)\rangle_{s-n} . \end{aligned} \tag{4.1}$$

Also, we obtain

$$\begin{aligned} \langle \partial_x^n w(t), w(t)\rangle_{s-n} &= 2\pi \textstyle\sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} \widehat{\partial_x^n w}(k) \cdot \overline{\widehat{w}(k)} \\ &= 2\pi \textstyle\sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} (ik)^n \widehat{w}(k) \cdot \overline{\widehat{w}(k)} \\ &= \underbrace{2\pi \textstyle\sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} k^n |\widehat{w}(k)|^2}_{b:=} , \end{aligned} \tag{4.2}$$

noting $(ik)^n = k^n$ when n is an even number multiple of four.

Now, we will prove that the series of the equality (4.2) is convergent. In effect, using the inequality: $|k|^n \leq |k|^{2n} = (|k|^2)^n \leq (1+|k|^2)^n$ and $w(t) \in H^s_{per}$ we have

$$\left| \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} k^n |\widehat{w}(k)|^2 \right| \leq \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} |k|^n |\widehat{w}(k)|^2$$

$$\leq \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-n} (1+|k|^2)^n |\widehat{w}(k)|^2$$

$$= \sum_{k=-\infty}^{+\infty} (1+k^2)^{s} |\widehat{w}(k)|^2 = \frac{1}{2\pi} \|w(t)\|^2_s < \infty .$$

Then the series (4.2) is convergent, that is,

$$\langle \partial_x^n w(t), w(t)\rangle_{s-n} = b \text{ , with } b \in R . \tag{4.3}$$

From (4.1), using $\partial_t w = -\partial_x^n w$ and the equality (4.3) we get

$$\begin{aligned} \partial_t \|w(t)\|^2_{s-n} &= 2Re\, \langle \partial_t w(t), w(t)\rangle_{s-n} \\ &= 2Re\, \langle -\partial_x^n w(t), w(t)\rangle_{s-n} \\ &= -2 \underbrace{Re\, \langle \partial_x^n w(t), w(t)\rangle_{s-n}}_{=b} \\ &= -2b \leq 0 \end{aligned}$$

since $b \geq 0$.

Therefore, $\|w(t)\|_{s-n}^2$ is not increasing. Then, $\|w(t)\|_{s-n}^2 \leq \|w(0)\|_{s-n}^2$, $\forall\, t \in [0,\infty)$. As $(\|w(t)\|_{s-n} - \|w(0)\|_{s-n})(\|w(t)\|_{s-n} + \|w(0)\|_{s-n}) \leq 0$, we have

$$\|w(t)\|_{s-n} \leq \|w(0)\|_{s-n} \leq \|w(0)\|_s \,, \forall t \in [0,\infty).$$

That is, $\|w(t)\|_{s-n} \leq \|\varphi\|_{s-n} \leq \|\varphi\|_s$, $\forall t \in [0,\infty)$.

■

Corollary 4.1
[Continuous dependence of the solution of (P_1)] Let u and v solutions of (P_1) with initial data ψ_1 and ψ_2 in H_{per}^s, respectively. Then the following assertions hold

$$\partial_t \|u(t) - v(t)\|_{s-n}^2 \leq 0$$

and

$$\|u(t) - v(t)\|_{s-n} \leq \|\psi_1 - \psi_2\|_{s-n} \leq \|\psi_1 - \psi_2\|_s \,, t \in [0,\infty). \quad (4.4)$$

Proof:
Define $w \coloneqq u - v$ then w satisfies

$$\left| \begin{array}{l} \partial_t w + \partial_x^n w = 0 \\ w(0) = \psi_1 - \psi_2 \,. \end{array} \right.$$

We conclude using the Theorem 4.1.

■

Corollary 4.2:
[Uniqueness of solution of (P_1)] The problem (P_1) has a unique solution.

Proof:
In effect, let u and v solutions of (P_1) with the same initial data, that is $\psi_1 = \psi_2 = \psi$. From (4.4) we obtain $\|u(t) - v(t)\|_{s-n} \leq \|0\|_s = 0$. Then, $\|u(t) - v(t)\|_{s-n} = 0$. So, $u(t) = v(t), \forall t \in R$, that is $u = v$.

■

Finally,

Remark 4.1:
When n is an even number not a multiple of four, the problem (P_1) has no solution.

CONCLUSIONS

By Fourier theory, we proved in detail the existence and uniqueness of solution to the model (P_1) when n is an even number multiple of four, as well as the continuous dependency of the solution respect to the initial data. Later on, we introduced the semigroup theory to rewrite the solution of problem (P_1) by this theory, making it much more fine. We used semigroups theory and get important results of existence and approximation. Finally, we demonstrate the dissipative property of the homogeneous problem, which allowed us to deduce the continuous dependence (with respect to the initial data) and uniqueness solution of (P_1).

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