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SERVICES FOR SCIENCE AND EDUCATION

Existence of the Local Solution of a Non-Homogeneous Schrödinger Type Equation

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ABSTRACT

In this article, we prove that initial value problem associated to the Schrödinger type non homogeneous equation in periodic Sobolev spaces has a local solution in [0, T] with T > 0, and the solution has continuous dependence with respect to the initial data and the non homogeneous part of the problem. We do this in a intuitive way using Fourier theory and introducing a C_o- group inspired by the work of Iorio [1] and Santiago [7]. Also, we prove the uniqueness solution of the Schrödinger type homogeneous equation, using its conservative property, inspired by the work of Iorio [1] and Santiago [6]. Finally, we study its generalization to n-th order equation.

Keywords: Uniqueness solution, Schrödinger type equation, non homogeneous equation, n-th order equation, periodic Sobolev spaces, Fourier Theory, calculus in Banach spaces.

INTRODUCTION

First, we want to comment that from Theorem 3.1 in [5], we have that the homogeneous problem is globally well posed and, in addition to the equality (3.2) in [5], we have the continuous dependence of the solution of homogeneous problem respect to the initial data.

In this work, in Theorem 3.2 we will prove the existence and uniqueness of the local solution for the non homogeneous problem and from inequality (3.8) we will get the continuous dependence of the solution with respect to the initial data and respect to the non homogeneous part.

Thus, in both homogeneous and non homogeneous cases, the estimatives are made from the explicit form of the solution, that is, by applying the Fourier transform to the respective equation. Another result in this work is about the conservative property of the homogeneous problem and some estimates of it, using differential calculus in H^s_{per} . This is included in Theorem 3.3 which we will develop in subsection 3.2. So, using Theorem 3.3, we deduce the results of continuous dependence and uniqueness of solution for homogeneous problem.

Finally, we get to generalize the results obtained.

We cite some works about Schrödinger equation [1], [5] and for dissipative properties of systems [2].

Our article is organized as follows. In section 2, we indicate the methodology used and cite the references used. In section 3, we proved the main results for the non homogeneous Schrödinger type equation. Also, in this section we manage to generalize the results to the n-th order equation.

Finally, in section 4, we give the conclusions of our study.

METHODOLOGY

As a theoretical framework in this article we use the existence and regularity results of [5]. Also, we use the references [1], [5], [6], [7], [8], [9] and [3] for the Fourier theory in periodic Sobolev spaces, and differential and integral calculus in Banach spaces.

MAIN RESULTS

First, using the Fourier transform, we will prove that the non homogeneous problem has a unique solution and it continuously depends respect to the initial data and the non homogeneity in compact intervals.

Second, we will study the uniqueness of the solution for homogeneous case using another technique that involves the conservative property of the problem.
Finally, we will get to generalize the results to n-th order equation.

The Non Homogeneous Problem (Q_3^F) is Locally Well Posed

Theorem 3.1:
Let s a fixed real number, $F \in C\big([0,T], H^s_{per}\big)$, where T>0, $\{S(t)\}_{t\in R}$ the unitary group of class C_o of homogeneous case (F = 0), introduced in the Theorem 4.1 from [5], and

$$u_p(t) := \int_0^t S(t-\tau)F(\tau)d\tau.$$

Then $u_p \in C\big([0,T], H^s_{per}\big) \cap C^1\big([0,T], H^{s-2}_{per}\big)$ and satisfies

$$\left| \begin{array}{c} \partial_t u_p(t) - i\mu\partial_x^2 u_p(t) + i\alpha\, u_p(t) = F(t) \ \in \ H^{s-2}_{per} \\ u_p(0) = 0 \end{array} \right. \tag{3.1}$$

with the derivative given by

$$\lim_{h\to 0} \left\| \frac{u_p(t+h)-u_p(t)}{h} - i\mu\partial_x^2 u_p(t) + i\alpha\, u_p(t) - F(t) \right\|_{s-2} = 0. \tag{3.2}$$

Proof:
We remark that $S(t-\tau)F(\tau) \in \ H^s_{per}, \forall\tau \in (0,t)$ and $\tau \to S(t-\tau)F(\tau)$ is continuous in [0, t] then exists $\underbrace{\int_0^t S(t-\tau)F(\tau)d\tau}_{u_p(t)=} \in H^s_{per}$.

Now, we will prove $u_p \in C([0,T], H^s_{per})$, that is, $\| u_p(t+h) - u_p(t) \|_s \to 0$ when $h \to 0$.

Let $h > 0$

$$\begin{aligned} u_p(t+h) - u_p(t) &= \int_0^{t+h} S(t+h-\tau)F(\tau)d\tau - \int_0^{t} S(t-\tau)F(\tau)d\tau \\ &= \int_0^{t} \{S(t+h-\tau) - S(t-\tau)\}F(\tau)d\tau \\ &\quad + \int_t^{t+h} S(t+h-\tau)F(\tau)d\tau \end{aligned}$$

taking the norm $\|\cdot\|_s$ we obtain

$$\begin{aligned} \left\|u_p(t+h) - u_p(t)\right\|_s &\leq \underbrace{\left\|\int_0^{t} \{S(t+h-\tau) - S(t-\tau)\}F(\tau)d\tau\right\|_s}_{I_1:=} \\ &\quad + \underbrace{\left\|\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau\right\|_s}_{I_2:=} . \end{aligned}$$

Using the M-Test of Weierstrass we get

$$\begin{aligned} I_1 &\leq \int_0^{t} \|S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)\|_s \, d\tau \\ &\leq \frac{\epsilon}{2T} \cdot \int_0^{t} d\tau = \frac{\epsilon t}{2T} \leq \frac{\epsilon T}{2T} = \frac{\epsilon}{2} \end{aligned}$$

since ∃δ > 0 such that:

$$\text{if } |h| < \delta \text{ then } \|S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)\|_s < \frac{\epsilon}{2T}, \forall \tau \in (0,t).$$

Using the mean value Theorem in H^s_{per} we obtain

$$\frac{1}{h} \cdot \int_t^{t+h} S(t+h-\tau)F(\tau)d\tau \to S(0)F(t) = F(t)$$

when $h \to 0$. Then

$$\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau = \underbrace{h}_{\to 0} \cdot \underbrace{\frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau}_{\to F(t)} \to 0$$

when $h \to 0$. So, we have

$$I_2 = \left\| \int_t^{t+h} S(t+h-\tau)F(\tau)d\tau \right\|_s \to 0$$

when $h \to 0$. That is $I_2 < \frac{\epsilon}{2}$ whenever $|h| < \delta^*$.

Therefore, $\left\| u_p(t+h) - u_p(t) \right\|_s \leq I_1 + I_2 < \epsilon$ whenever $|h| < \min\{\delta, \delta^*\}$.

From definition of $u_p(t)$ we have $u_p(0) = 0$.

Now, we will prove that $\exists\ \partial_t u_p(t)$ in H_{per}^{s-2} . In effect

$$\frac{u_p(t+h) - u_p(t)}{h}$$

$$= \frac{1}{h}\left\{\int_0^{t+h} S(t+h-\tau)F(\tau)d\tau - \int_0^t S(t-\tau)F(\tau)d\tau\right\}$$

$$= \frac{1}{h}\left\{\int_0^t \{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)\}\, d\tau + \int_t^{t+h} S(t+h-\tau)F(\tau)d\tau\right\}$$

$$= \frac{1}{h}\int_0^t \{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)\}\, d\tau + \frac{1}{h}\int_t^{t+h} S(t+h-\tau)F(\tau)d\tau.$$

Using the mean value Theorem in H_{per}^{s-2} we obtain

$$\frac{1}{h}\cdot \int_t^{t+h} S(t+h-\tau)F(\tau)d\tau \to S(0)F(t) = F(t) \tag{3.3}$$

when $h \to 0$.

Since

$$\frac{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)}{h}$$

converges uniformly to $\partial_t\{S(t-\tau)F(\tau)\}$ in H_{per}^{s-2} $\forall \tau \in [0,t]$, we obtain that

$$\int_0^t \frac{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)}{h}\, d\tau$$

converges to $\int_0^t \partial_t\{S(t-\tau)F(\tau)\}\, d\tau$ when $h \to 0$.

Now, we remark that $\partial_t\{S(t-\tau)F(\tau)\} = (i\mu\partial_x^2)S(t-\tau)F(\tau) - i\alpha S(t-\tau)F(\tau)$ in H_{per}^{s-2} .

Since $(i\mu\partial_x^2)$ is a closed operator, then

$$\int_0^t (i\mu\partial_x^2)S(t-\tau)F(\tau)d\tau = (i\mu\partial_x^2)\int_0^t S(t-\tau)F(\tau)\, d\tau.$$

Therefore,

$$\int_0^t \partial_t\{S(t-\tau)F(\tau)\}\; d\tau = (i\mu\partial_x^2)\underbrace{\int_0^t S(t-\tau)F(\tau)\, d\tau}_{u_p(t)=} - i\alpha \underbrace{\int_0^t S(t-\tau)F(\tau)\, d\tau}_{u_p(t)=}.$$

That is,

$$\int_0^t \frac{S(t+h-\tau)F(\tau) - S(t-\tau)F(\tau)}{h}\; d\tau \;\to\; (i\mu\partial_x^2)u_p(t) - i\alpha u_p(t) \tag{3.4}$$

when $h \to 0$.

Finally, in H_{per}^{s-2} , using (3.3) and (3.4) we get

$$\exists \underbrace{\lim_{h\to 0}\frac{u_p(t+h)-u_p(t)}{h}}_{\partial_t u_p(t)=} = (i\mu\partial_x^2)u_p(t) - i\alpha u_p(t) + F(t)\,.$$

Using the inequality $\|\partial_x^m f\|_{s-m} \leq \|f\|_s$,$\forall m \in Z^+$,$\forall\, f \in H_{per}^s$, we obtain

$$\left\|\partial_t u_p(t+h) - \partial_t u_p(t)\right\|_{s-2}$$

$$\begin{aligned}
&= \left\|(i\mu\partial_x^2)u_p(t+h) - i\alpha u_p(t+h) + F(t+h) - \left\{(i\mu\partial_x^2)u_p(t) - i\alpha u_p(t) + F(t)\right\}\right\|_{s-2}\\
&\leq \|F(t+h)-F(t)\|_{s-2} + \left\|(i\mu\partial_x^2)\{\,u_p(t+h)-u_p(t)\}\right\|_{s-2} + \left\|i\alpha\{\,u_p(t+h)-u_p(t)\}\right\|_{s-2}\\
&= \|F(t+h)-F(t)\|_{s-2} + |\mu|\left\|\partial_x^2\{\,u_p(t+h)-u_p(t)\}\right\|_{s-2} + |\alpha|\,\left\|\{\,u_p(t+h)-u_p(t)\}\right\|_{s-2}\\
&\leq \|F(t+h)-F(t)\|_{s-2} + |\mu|\left\|\,u_p(t+h)-u_p(t)\right\|_s + |\alpha|\left\|\,u_p(t+h)-u_p(t)\right\|_{s-2}.
\end{aligned}$$

So, since $u_p(t) \in H_{per}^s \;\subset\; H_{per}^{s-1} \subset\; H_{per}^{s-2}$, we have

$$\begin{aligned}
&\left\|\partial_t u_p(t+h) - \partial_t u_p(t)\right\|_{s-2}\\
&\leq \|F(t+h)-F(t)\|_{s-2} + (|\mu|+|\alpha|)\left\|\,u_p(t+h)-u_p(t)\right\|_s.
\end{aligned}$$

Now, since $F: [0,T] \to H_{per}^{s-2}$ and $u_p: [0,T] \to H_{per}^{s}$ are continuous, then $\partial_t u_p: [0,T] \to H_{per}^{s-2}$ is continuous for $t \in [0,T]$, that is, $\partial_t u_p \in C([0,T], H_{per}^{s-2})$.

Therefore, $u_p \in C([0,T], H_{per}^{s}) \cap C^1([0,T], H_{per}^{s-2})$.

Theorem 3.2

Let s a fixed real number, $\varphi \in H_{per}^{s}$, $F \in C\big([0,T], H_{per}^{s}\big)$, where T>0 and $\{S(t)\}_{t\in R}$ the unitary group of class C_o in H_{per}^{s} as in Theorem 3.1, then

The function

$$u^F(t) := S(t)\varphi + \underbrace{\int_0^t S(t-\tau)F(\tau)\,d\tau}_{u_p(t)=}, t \in [0,T] \tag{3.5}$$

belongs to $C([0,T], H_{per}^{s}) \cap C^1([0,T], H_{per}^{s-2})$ and $u^F(t)$ is the unique solution of

$$(Q_3^F) \quad \left| \begin{array}{c} u_t - i\mu\partial_x^2 u + i\alpha u = F(t) \in H_{per}^{s-2} \\ u(0) = \varphi \end{array} \right. \tag{3.6}$$

with the derivative given by

$$\lim_{h\to 0} \left\| \frac{u(t+h)-u(t)}{h} - i\mu\partial_x^2 u(t) + i\alpha u(t) - F(t) \right\|_{s-2} = 0. \tag{3.7}$$

The map $\psi \to u$ is continuous in the following sense. Let $\psi_j \in H_{per}^{s}$, $F_j \in C\big([0,T], H_{per}^{s}\big)$ $para\ j = 1{,}2$ then u_1 and u_2 the corresponding solutions to initial data ψ_1 and ψ_2, and with non homogeneity F_1 and F_2 respectively, satisfy

$$\|u_1(t) - u_2(t)\|_s \leq \|\psi_1 - \psi_2\|_s + T\|F_1 - F_2\|_{\infty,s}, t \in [0,T], \tag{3.8}$$

$$\underbrace{sup_{t\in[0,T]}\|u_1(t) - u_2(t)\|_s}_{\|u_1-u_2\|_{\infty,s}=} \leq \|\psi_1 - \psi_2\|_s + T\|F_1 - F_2\|_{\infty,s} \tag{3.9}$$

$$\|\partial_t u_1(t) - \partial_t u_2(t)\|_{s-2}$$

$$\begin{aligned} &\leq (|\mu| + |\alpha|)\|u_1(t) - u_2(t)\|_s + \|F_1 - F_2\|_{\infty,s-2}, t \in [0,T], \\ &\leq (|\mu| + |\alpha|)\|u_1 - u_2\|_{\infty,s} + \|F_1 - F_2\|_{\infty,s} \\ &\leq (|\mu| + |\alpha|)\|\psi_1 - \psi_2\|_{\infty,s} + \big((|\mu| + |\alpha|)T + 1\big)\|F_1 - F_2\|_{\infty,s} \end{aligned} \tag{3.10}$$

where we have used the notation

$$\|h\|_{\infty,r} := sup_{t\in[0,T]}\|h(t)\|_r, h \in C\big([0,T], H_{per}^{r}\big). \tag{3.11}$$

Proof:
We work the following steps.

Let $u(t) \coloneqq u^F(t) = S(t)\varphi + u_p(t)$, we will prove that $u \in C\big([0,T], H^s_{per}\big) \cap C^1\big([0,T], H^{s-2}_{per}\big)$. In effect, as $S(\cdot)\varphi \in C\big([0,T], H^s_{per}\big)$ and $u_p(\cdot) \in C\big([0,T], H^s_{per}\big)$ then $u(\cdot) = S(\cdot)\varphi + u_p(\cdot) \in C\big([0,T], H^s_{per}\big)$.

Moreover, as

$$S(\cdot)\varphi \in C^1\big([0,+\infty), H^{s-2}_{per}\big) \text{ and } u_p(\cdot) \in C^1\big([0,T], H^{s-2}_{per}\big)$$

then

$$u(\cdot) = S(\cdot)\varphi + u_p(\cdot) \in C^1\big([0,T], H^{s-2}_{per}\big).$$

We will prove that u is the solution of (Q_3^F). In effect, we know that $\exists \partial_t S(t)\varphi$ and $\exists \partial_t u_p(t)$ in H^{s-2}_{per} , then

$$\begin{aligned} \partial_t u(t) &= \partial_t \underbrace{S(t)\varphi}_{u_h(t)\coloneqq} + \partial_t u_p(t) \\ &= i\mu\partial_x^2 u_h(t) - i\alpha u_h(t) + i\mu\partial_x^2 u_p(t) - i\alpha u_p(t) + F(t) \\ &= i\mu\partial_x^2\big\{u_h(t) + u_p(t)\big\} - i\alpha\big\{u_h(t) + u_p(t)\big\} + F(t) \\ &= i\mu\partial_x^2 u(t) - i\alpha u(t) + F(t) \end{aligned}$$

in H^{s-2}_{per} , where $u_h(\cdot)$ is solution of the homogeneous equation ($Q_3 \coloneqq Q_3^0$).

Also, $u(0) = u_h(0) + u_p(0) = \varphi + 0 = \varphi$.

Let $\psi_j \in H^s_{per}$ and $F_j \in C\big([0,T], H^s_{per}\big)$ for $j = 1,2$, then

$$u_j(t) = S(t)\psi_j + \int_0^t S(t-\tau)F_j(\tau)d\tau$$

is solution of $(Q_3^{F_j})$ with initial data $u_j(0) = S(0)\psi_j = \psi_j$, for $j = 1,2$.

Then

$$u_1(t) - u_2(t) = S(t)\{\psi_1 - \psi_2\} + \int_0^t S(t-\tau)\{F_1(\tau) - F_2(\tau)\}d\tau.$$

From where we obtain, for t<T:

$$\|u_1(t) - u_2(t)\|_s \leq \|S(t)\{\psi_1 - \psi_2\}\|_s + \int_0^t \|S(t-\tau)\{F_1(\tau) - F_2(\tau)\}\|_s d\tau$$

$$\begin{aligned} &= \|\psi_1 - \psi_2\|_s + \int_0^t \|F_1(\tau) - F_2(\tau)\|_s d\tau \\ &\leq \|\psi_1 - \psi_2\|_s + sup_{\tau\in[0,T]}\|F_1(\tau) - F_2(\tau)\|_s \int_0^t d\tau \\ &\leq \|\psi_1 - \psi_2\|_s + T \cdot sup_{\tau\in[0,T]}\|F_1(\tau) - F_2(\tau)\|_s. \end{aligned}$$

Therefore,

$$sup_{\tau\in[0,T]}\|u_1(t) - u_2(t)\|_s \leq \|\psi_1 - \psi_2\|_s + T \cdot sup_{\tau\in[0,T]}\|F_1(\tau) - F_2(\tau)\|_s.$$

On the other hand, in H_{per}^{s-2} we have

$$\begin{aligned} \partial_t u_j(t) &= \partial_t u_{h,j}(t) + \partial_t u_{p,j}(t) \\ &= [i\mu\partial_x^2]u_{h,j}(t) - i\alpha u_{h,j}(t) + [i\mu\partial_x^2]u_{p,j}(t) - i\alpha u_{p,j}(t) + F_j(t) \\ &= [i\mu\partial_x^2]u_j(t) - i\alpha u_j(t) + F_j(t)\,. \end{aligned}$$

for $j = 1,2$. So,

$$\partial_t u_1(t) - \partial_t u_2(t) = [i\mu\partial_x^2]\{u_1(t) - u_2(t)\} - i\alpha\{u_1(t) - u_2(t)\} + \{F_1(t) - F_2(t)\}\,.$$

Taking norm, we obtain

$$\begin{gathered} \|\partial_t u_1(t) - \partial_t u_2(t)\|_{s-2} \\ = \|[i\mu\partial_x^2]\{u_1(t) - u_2(t)\} - i\alpha\{u_1(t) - u_2(t)\} + \{F_1(t) - F_2(t)\}\|_{s-2} \\ \leq \|[i\mu\partial_x^2]\{u_1(t) - u_2(t)\}\|_{s-2} + \|\alpha\{u_1(t) - u_2(t)\}\|_{s-2} + \|F_1(t) - F_2(t)\|_{s-2}. \end{gathered}$$

Using

$$\|\partial_x^2 f\|_{s-2} \leq \|f\|_s\,, \forall\, f \in H_{per}^s \text{ and } H_{per}^s \subset H_{per}^{s-1} \subset H_{per}^{s-2}\,,$$

we get

$$\begin{gathered} \|\partial_t u_1(t) - \partial_t u_2(t)\|_{s-2} \\ \leq |\mu|\|u_1(t) - u_2(t)\|_s + |\alpha|\|u_1(t) - u_2(t)\|_{s-2} + \|F_1(t) - F_2(t)\|_{s-2} \\ \leq (|\mu| + |\alpha|)\|u_1(t) - u_2(t)\|_s + sup_{t\in[0,T]}\|F_1(t) - F_2(t)\|_{s-2} \\ \leq (|\mu| + |\alpha|)\|u_1 - u_2\|_{\infty,s} + \|F_1 - F_2\|_{\infty,s} \\ \leq (|\mu| + |\alpha|)\|\psi_1 - \psi_2\|_s + ((|\mu| + |\alpha|)T + 1)\|F_1 - F_2\|_{\infty,s}. \end{gathered}$$

■

Remark 3.1

Inequality (3.8) says that the solution of the non homogeneous problem (Q_3^F) continuously depends on the initial data and the non homogeneity F, in compact intervals.

Corollary 3.1:
The problem (Q_3^F) has a unique solution.

Proof:
This follows by applying inequality (3.8) with $\psi_1 = \psi_2 = \varphi$ and $F_1 = F_2 = F$.

■

Corollary 3.2:
The unique solution of (Q_3^F) is

$$u(t) = \sum_{k=-\infty}^{+\infty} e^{-i\mu k^2 t} e^{-i\alpha t} \hat{\varphi}(k) \varphi_k + \int_0^t \sum_{k=-\infty}^{+\infty} e^{-i\mu k^2 (t-\tau)} e^{-i\alpha(t-\tau)} \hat{F}(k,\tau) \varphi_k \, d\tau ,$$

where $\varphi_k(x) := e^{ikx}$ for x ∈ R.

Conservative Property of The Homogeneous Problem
Let $s \in R, \mu > 0, \alpha > 0$ and the homogeneous problem

$$(Q_3) \left| \begin{array}{c} w \in C(R, H^s_{per}) \cap C^1(R, H^{s-2}_{per}) \\ \partial_t w - i\mu \partial_x^2 w + i\alpha w = 0 \in H^{s-2}_{per} \\ w(0) = \varphi \in H^s_{per} \end{array} \right.$$

Theorem 3.3:
Let w the solution of (Q_3) with initial data $\varphi \in H^s_{per}$ then we obtain the following results:

$$\partial_t \|w(t)\|^2_{s-2} = 0.$$

$$\|w(t)\|_{s-2} = \|\varphi\|_{s-2} \leq \|\varphi\|_s , \forall t \in R .$$

Proof:
As $H^s_{per} \subset H^{s-2}_{per}$ then the following expressions: $\langle \partial_t w, w \rangle_{s-2}$ and $\langle w, \partial_t w \rangle_{s-2}$ are well defined.

So, we have

$$\begin{aligned} \partial_t \|w(t)\|^2_{s-2} &= \partial_t \langle w(t), w(t) \rangle_{s-2} \\ &= \langle \partial_t w(t), w(t) \rangle_{s-2} + \langle w(t), \partial_t w(t) \rangle_{s-2} \\ &= 2Re \, \langle \partial_t w(t), w(t) \rangle_{s-2} \, . \; (3.12) \end{aligned}$$

Also, we obtain

$$\begin{aligned}
\langle \partial_x^2 w(t), w(t)\rangle_{s-2} &= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-2} \widehat{\partial_x^2 w}(k) \cdot \overline{\widehat{w}(k)} \\
&= 2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-2} (ik)^2 \widehat{w}(k) \cdot \overline{\widehat{w}(k)} \\
&= -\underbrace{2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-2} k^2 |\widehat{w}(k)|^2}_{b:=} . \qquad (3.13)
\end{aligned}$$

Now, we will prove that the series of the equality (3.13) is convergent. In effect, using the inequality: $|k|^2 \leq |k|^4 = (|k|^2)^2 \leq (1+|k|^2)^2$ and $w(t) \in H^s_{per}$ we have

$$\begin{aligned}
\left| \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-2} k^2 |\widehat{w}(k)|^2 \right| &\leq \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-2} |k|^2 |\widehat{w}(k)|^2 \\
&\leq \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-2} (1+|k|^2)^2 |\widehat{w}(k)|^2 \\
&= \sum_{k=-\infty}^{+\infty} (1+k^2)^{s} |\widehat{w}(k)|^2 = \frac{1}{2\pi} \|w(t)\|_s^2 < \infty .
\end{aligned}$$

Then the series (3.13) is convergent, that is,

$$\langle \partial_x^2 w(t), w(t)\rangle_{s-2} = -b \text{ , with } b \in R^+ . \qquad (3.14)$$

As $\widehat{\partial_x w}(k) = ik\widehat{w}(k)$ and $\overline{\widehat{\partial_x w}(k)} = -ik\overline{\widehat{w}(k)}$, then their product is

$$\widehat{\partial_x w}(k) \cdot \overline{\widehat{\partial_x w}(k)} = k^2 \widehat{w}(k)\overline{\widehat{w}(k)} = k^2 |\widehat{w}(k)|^2. \qquad (3.15)$$

Substituting (3.15) in (3.13) we obtain

$$\begin{aligned}
\langle \partial_x^2 w(t), w(t)\rangle_{s-2} &= -2\pi \sum_{k=-\infty}^{+\infty} (1+k^2)^{s-2} \, \widehat{\partial_x w}(k) \cdot \overline{\widehat{\partial_x w}(k)} \\
&= -< \partial_x w, \partial_x w >_{s-2} \\
&= -\|\partial_x w\|_{s-2}^2 \leq 0. \qquad (3.16)
\end{aligned}$$

Obviously $\|\partial_x w\|_{s-2} < \infty$ since $\|\partial_x w\|_{s-1} < \infty$ and $\|\partial_x w\|_{s-2} \leq \|\partial_x w\|_{s-1}$.

From (3.12), using $\partial_t w = i\mu \partial_x^2 w - i\alpha w$ and the equality (3.16) we get

$$\begin{aligned}\partial_t\|w(t)\|_{s-2}^2 &= 2Re\,\langle\partial_t w(t), w(t)\rangle_{s-2}\\ &= 2Re\,\langle i\mu\partial_x^2 w(t) - i\alpha w(t), w(t)\rangle_{s-2}\\ &= 2\mu Re\{\,i\langle\partial_x^2 w(t), w(t)\rangle_{s-2}\} - 2\alpha Re\{i < w(t).w(t) >_{s-2}\}\\ &= 2\mu Re\{-\,i\|\partial_x w\|_{s-2}^2\} - 2\alpha Re\{\,i\|w\|_{s-2}^2\} = 0 - 0 = 0\,.\end{aligned}$$

Therefore, $\|w(t)\|_{s-2}^2$ is a constant. Then, $\|w(t)\|_{s-2}^2 = \|w(0)\|_{s-2}^2$, $\forall\, t \in R$.

As

$$(\|w(t)\|_{s-2} - \|w(0)\|_{s-2})(\|w(t)\|_{s-2} + \|w(0)\|_{s-2}) = 0\,,$$

we have

$$\|w(t)\|_{s-2} = \|w(0)\|_{s-2} \leq \|w(0)\|_s\,, \forall t\ \in R.$$

That is, $\|w(t)\|_{s-2} = \|\varphi\|_{s-2} \leq \|\varphi\|_s$, $\forall t\ \in R$.

■

Corollary 3.3:
[Continuous dependence of the solution of (Q_3)] Let u and v solutions of (Q_3) with initial data ψ_1 and ψ_2 in H_{per}^s , respectively. Then the following assertions hold

$$\partial_t\|u(t) - v(t)\|_{s-2}^2 = 0$$

and

$$\|u(t) - v(t)\|_{s-2} = \|\psi_1 - \psi_2\|_{s-2} \leq \|\psi_1 - \psi_2\|_s\,, t \in R. \tag{3.17}$$

Proof:
Define $w := u - v$ then w satisfies

$$\left|\begin{array}{c}\partial_t w - i\mu\partial_x^2 w + i\alpha w = 0\\ w(0) = \psi_1 - \psi_2\,.\end{array}\right.$$

We conclude using the Theorem 3.3.

■

Corollary 3.4:
[Uniqueness of solution of (Q_3)] The problem (Q_3) has a unique solution.

Proof:
In effect, let u and v solutions of (Q_3) with the same initial data, that is $\psi_1 = \psi_2 = \psi$.

From (3.17) we obtain $\|u(t)-v(t)\|_{s-2} \le \|0\|_s = 0$. Then, $\|u(t)-v(t)\|_{s-2} = 0$. So, $u(t) = v(t), \forall t \in R$, that is $u = v$.

■

Generalization of Results for Generalized Schrödinger Type Equation

Theorem 3.4 Let s a fixed real number, n an even number not multiple of four, $F \in C\big([0,T], H^s_{per}\big)$, where T>0, $\{\mathcal{T}(t)\}_{t\in R}$ the unitary group of class C_o of homogeneous case (F = 0), introduced in the Theorem 4.1 from [4], and

$$u_p(t) := \int_0^t \mathcal{T}(t-\tau)F(\tau)d\tau.$$

Then $u_p \in C\big([0,T], H^s_{per}\big) \cap C^1\big([0,T], H^{s-n}_{per}\big)$ and satisfies

$$\left| \begin{array}{c} \partial_t u_p - i\mu\partial_x^n u_p + i\alpha u_p = F(t) \; \in \; H^{s-n}_{per} \\ u_p(0) = 0 \end{array} \right. \tag{3.18}$$

with the derivative given by

$$\lim_{h\to 0} \left\| \frac{u_p(t+h)-u_p(t)}{h} - i\mu\partial_x^n u_p + i\alpha u_p - F(t) \right\|_{s-n} = 0 \tag{3.19}$$

Theorem 3.5:

Let s a fixed real number, $\varphi \in H^s_{per}$, n an even number not multiple of four , $F \in C\big([0,T], H^s_{per}\big)$, where T>0, and $\{\mathcal{T}(t)\}_{t\in R}$ the unitary group of class C_o in H^s_{per} as in Theorem 3.4, then

The function

$$u^F(t) \coloneqq \mathcal{T}(t)\varphi + \underbrace{\int_0^t \mathcal{T}(t-\tau)F(\tau)\,d\tau}_{u_p(t)=}, t \in [0,T] \tag{3.20}$$

belongs to $C([0,T], H^s_{per}) \cap C^1([0,T], H^{s-n}_{per})$ and

$u^F(t)$ is the unique solution of

$$(Q^F_{n+1}) \quad \left| \begin{array}{c} u_t - i\mu\partial_x^n u + i\alpha u = F(t) \in H^{s-n}_{per} \\ u(0) = \varphi \end{array} \right. \tag{3.21}$$

with the derivative given by

$$\lim_{h\to 0}\left\|\frac{u(t+h)-u(t)}{h}-i\mu\partial_x^n u+i\alpha u-F(t)\right\|_{s-n}=0\,. \tag{3.22}$$

The map $\psi \to u$ is continuous in the following sense. Let $\psi_j \in H^s_{per}$, $F_j \in C\big([0,T],H^s_{per}\big), j = 1,2$ *then* u_1 and u_2 the corresponding solutions to initial data ψ_1 and ψ_2, and with non homogeneity F_1 and F_2 respectively, satisfy

$$\|u_1(t)-u_2(t)\|_s \quad\leq\quad \|\psi_1-\psi_2\|_s+T\|F_1-F_2\|_{\infty,s}, t\in[0,T], \tag{3.23}$$

$$\underbrace{sup_{t\in[0,T]}\|u_1(t)-u_2(t)\|_s}_{\|u_1-u_2\|_{\infty,s}=} \quad\leq\quad \|\psi_1-\psi_2\|_s+T\|F_1-F_2\|_{\infty,s} \tag{3.24}$$

$$\begin{aligned}&\|\partial_t u_1(t)-\partial_t u_2(t)\|_{s-n}\\ &\leq (|\boldsymbol{\mu}|+|\boldsymbol{\alpha}|)\|u_1(t)-u_2(t)\|_s+\ \|F_1-F_2\|_{\infty,s-n}\,, t\in[0,T],\\ &\leq (|\boldsymbol{\mu}|+|\boldsymbol{\alpha}|)\|u_1-u_2\|_{\infty,s}+\|F_1-F_2\|_{\infty,s}\\ &\leq (|\boldsymbol{\mu}|+|\boldsymbol{\alpha}|)\|\psi_1-\psi_2\|_{\infty,s}+\{(|\boldsymbol{\mu}|+|\boldsymbol{\alpha}|)T+1\}\|F_1-F_2\|_{\infty,s}\end{aligned} \tag{3.25}$$

where we have used the notation

$$\|h\|_{\infty,r} \coloneqq\ sup_{t\in[0,T]}\|h(t)\|_r, h\ \in C\big([0,T],H^r_{per}\big). \tag{3.26}$$

Remark 3.2:
Inequality (3.23) says that the solution of the non homogeneous problem (Q^F_{n+1}) continuously depends on the initial data and the non homogeneity F, in compact intervals.

Corollary 3.5:
The problem (Q^F_{n+1}) has a unique solution.

Proof:
This follows by applying inequality (3.23) with $\psi_1=\psi_2=\varphi$ and $F_1=F_2=F$.

■

Corollary 3.6:
The unique solution of (Q^F_{n+1}) is

$$u(t)=\sum_{k=-\infty}^{+\infty} e^{-i\mu k^n t}e^{-i\alpha t}\hat{\varphi}(k)\varphi_k$$
$$+\int_0^t\sum_{k=-\infty}^{+\infty} e^{-i\mu k^n (t-\tau)}e^{-i\alpha(t-\tau)}\hat{F}(k,\tau)\varphi_k\ d\tau\,,$$

where $\varphi_k(x) := e^{ikx}$ for x ∈ R.

Conservative Property of The Problem (Q_{n+1})

Let $s \in R$, n an even number not multiple of four and the homogeneous problem

$$(Q_{n+1}) \left| \begin{array}{c} w \in C\left(R, H_{per}^{s}\right) \cap C^{1}(R, H_{per}^{s-n}) \\ \partial_t w - i\mu \partial_x^n w + i\alpha w = 0 \in H_{per}^{s-n} \\ w(0) = \varphi \in H_{per}^{s} . \end{array} \right.$$

Theorem 3.6:

Let n is an even number not multiple of four and w the solution of (Q_{n+1}) with initial data $\varphi \in H_{per}^{s}$ then we obtain the following results:

$$\partial_t \|w(t)\|_{s-n}^2 = 0.$$

$$\|w(t)\|_{s-n} = \|\varphi\|_{s-n} \leq \|\varphi\|_s , \forall t \in R .$$

Proof:

The proof is analogous to the proof of Theorem 3.3, noting that $(ik)^n = -k^n, \forall k \in Z$ when n is an even number not multiple of four and $k^{n-2} \geq 1, \forall k \in Z - \{0\}$ when n is an even number.

■

Corollary 3.7:

[Continuous dependence of the solution of (Q_{n+1})] Let u and v solutions of (Q_{n+1}) with initial data ψ_1 and ψ_2 in H_{per}^{s} , respectively. Then the following assertions hold

$$\partial_t \|u(t) - v(t)\|_{s-n}^2 = 0$$

and

$$\|u(t) - v(t)\|_{s-n} = \|\psi_1 - \psi_2\|_{s-n} \leq \|\psi_1 - \psi_2\|_s , t \in R. \qquad (3.27)$$

Proof:

Define $w := u - v$ then w satisfies

$$\left| \begin{array}{c} \partial_t w - i\mu \partial_x^n w + i\alpha w = 0 \\ w(0) = \psi_1 - \psi_2 . \end{array} \right.$$

We conclude using the Theorem 3.6.

■

Corollary 3.8:

[Uniqueness of solution of (Q_{n+1})] The problem (Q_{n+1}) has a unique solution.

Proof:

In effect, let u and v solutions of (Q_{n+1}) with the same initial data, that is $\psi_1 = \psi_2 = \psi$.

From (3.27) we obtain $\|u(t) - v(t)\|_{s-n} \leq \|0\|_s = 0$. Then, $\|u(t) - v(t)\|_{s-n} = 0$. So, $u(t) = v(t), \forall t \in R$, that is $u = v$.

■

Remark 3.3:
Results analogous to Theorems 3.4 and 3.5 are obtained when n is an even number multiple of four, where the solution non-homogeneous is $V_*^F(t) := V(t)\varphi + V_p(t)$ with

$$\mathrm{V}(t)\varphi = \left[\left(e^{i\mu k^n t} e^{-i\alpha t} \hat{\varphi}(k)\right)_{k \in Z}\right]^{\vee}$$

for the initial data $\varphi \in H^s_{per}$ and $V_p(t) := \int_0^t V(t-\tau)F(\tau)d\,\tau$.

Remark 3.4:
Results analogous to theorem 3.6, Corollaries 3.7 and 3.8 are also obtained when n is an even number multiple of four, in this case, note that $(ik)^n = k^n$.

CONCLUSIONS

From our study of the Schrödinger type equation in periodic Sobolev spaces, we have obtained the following results:

1. In Theorem 3.1 we obtain a particular solution of (Q_3^F) using calculus in H^s_{per} and Sobolev inmersion.
2. In Theorem 3.2, using the Fourier transform, we proved that the non homogeneous problem (Q_3^F) is locally well posed in compacts, obtaining continuous dependence with respect to the initial data and the non homogeneity.
3. We showed the conservative property of the homogeneous problem, which allowed us to deduce the continuous dependence (with respect to the initial data) and uniqueness solution of (Q_3).
4. We obtained results analogous to Theorem 3.1, Theorem 3.2, Corollary 3.1, Corollary 3.2, Theorem 3.3, Corollary 3.3 and Corollary 3.4 for the n-th order Schrödinger type equation when n is an even number not multiple of four.
5. Finally, we gave some remarks about the n-th order Schrödinger type equation when n is an even number multiple of four.

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